Sparse polynomial chaos expansion for stability analysis of a clutch system with uncertain parameters

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Abstract

In the transmission systems of vehicles, unforced vibrations can be observed during the sliding phase of clutch engagement. These vibrations are due to frictional forces and may generate noise. Several studies have shown that the stability of such friction systems is highly sensitive to parameters (e.g. friction law, damping) which lead to signicant dispersion. Therefore, uncertain parameters must be considered in the stability analysis of a clutch system. In several studies of the literature, the usual generalized polynomial chaos (PC) expansion has been used to study the stability of a clutch system using non-intrusive techniques. However, non-intrusive techniques require a number of model evaluations (i.e. the computational cost) which can become prohibitive when the studied system has a large number of uncertain parameters. To remedy this problem, in this work, we use the sparse polynomial chaos expansion which has been recently developed in reliability domain. The method is compared to the reference Monte Carlo method and with the usual PC expansion in the context of the stability analysis of a clutch system. The results show that the use of the sparse PC allows a remarkable reduction of the computational cost by ensuring a high accuracy compared with the usual PC expansion.

Keywords: Stability; Vibration; Clutch; Friction system ; Sparse polynomial chaos; Regression methods

1 Introduction

Dry friction systems can develop dynamic instabilities related to the friction. In particular, in vehicles with manual transmission systems, these instabilities can be the cause of unwanted vibrations during the sliding phase of the clutch engagement. Several studies have been focused on the mechanisms responsible for these friction-induced vibrations [1]. The numerous mechanisms explaining the friction-induced vibration phenomenon are classified into two main families which are related to the tribological aspects of friction systems and to the geometrical and structural properties. The first family explains the instabilities by the variation of the friction coefficient with respect to the relative speed or by a higher static friction coefficient than the dynamic one. The stick-slip is a well-known phenomenon in this context [2]. The second family attributes the appearance of instabilities to the sprag-slip mechanism and more generally to the mode-coupling phenomenon. In this case, self-excited oscillations may occur even with a constant friction coefficient. Because of the speed range of the measured vibrations, high frequency of the self-excited phenomenon in clutch system cannot be related to stick-slip. Consequently, mode coupling instabilities due to the intrinsic structure of the system are more likely to be responsible for this phenomenon [3]. It is therefore necessary to compute the system eigenvalues which are complex functions because of the friction [4]. The analysis of the real parts sign of the eigenvalues allow to conclude about the stability of the system while the imaginary parts give the frequency of the corresponding mode.

It is known that the dynamic behavior of clutch systems is highly sensitive to design parameters, in particular to the friction coefficient and damping [1, 5]. In addition, the design process can lead to the dispersion of system parameters. Therefore, it is necessary to take into account uncertainties in the system parameters to ensure the robustness of the analysis of friction systems as clutch system and consequently the robustness of the design of this class of systems. The classic Monte Carlo approach which was used to reach this aim requires prohibitive computational cost.

As an alternative, the generalized Polynomial Chaos (gPC) and Multi-Element generalized Polynomial Chaos (ME-gPC) has been used in the past to take into account the uncertainties of the friction coefficient in the study of the dynamic behaviour of friction systems. With gPC [6, 7], when the number of uncertain parameters and the order of gPC increase, the number of simulation needed to estimate the PC coefficients becomes extremely large leading to a prohibitive computational cost. To solve this problem, the ME-gPC may be used. In particular, the ME-gPC had been applicated by Trinh et al. [8] for the stability analysis of a clutch system, the conclusion is that the use of the ME-gPC allows a reduction of the computation cost. However, the method is effective with a small number of uncertain parameters (up to 5) but for a higher number of uncertain parameters the computation cost become prohibitive again. To remedy this problem, one considers sparse Polynomial Chaos (sparse PC) expansion. This method is filled out by an iterative procedure allowing to build iteratively a sparse PC expansion while mastering the approximation error [9]. Sparse PC is very efficient for reducing computational cost in reliability analysis of static mechanical structures. This justifies the present work in which sparse PC strategies are used in the context stability analysis of a clutch system model.

The main aim of this study is to investigate the capacity of the sparse PC with Isotropic hyperbolic index sets to study the stability of a clutch system with an increasing number of uncertain parameters compared with the full PC expansion. The main task is to find a harmony between high accuracy and reasonable computational cost.

The paper is organized as follows. In section 2, the stability analysis of dynamical systems with the indirect Lyapounov method is recalled. The Polynomial Chaos theory is briefly presented in section 3 including the description of the gPC, ME-gPC and sparse PC. The section 4 presents the dynamical model of the clutch system under consideration in this study. Finally, the section 5 is dedicated to the comparison of the three polynomial chaos expansions for the stability analysis of the clutch system and to comment on the results.

2 Stability analysis of dynamic system with the indirect Lyapounov approach

Let us consider the motion equation of a nonlinear dynamic system

$$\dot{x} = f(x, u), \tag{1}$$

where x is the state vector of the system, f is a nonlinear function of x, and u is the uncertain parameter vector.

The *Hartman-Grobman theorem* (e.g., Wiggins [10], Chap. 3) states that in the vicinity of a hyperbolic equilibrium point[1], a nonlinear system has the same qualitative stability as does the corresponding linear system. Let the origin x_0 be an equilibrium point for the system (1). The Jacobian matrix of the system (1) is given by [10]

$$H(u) = \left. \frac{\partial f(x,u)}{\partial x} \right|_{x=x_0}. \tag{2}$$

According to the Lyapunov's indirect method, the stability of x_0 is analyzed by evaluating the eigenvalues $\lambda_i (i = 1, ..., n)$ of the matrix H:

1. The origin x_0 is asymptotically stable if Re($\lambda_i < 0$), $\forall i \in (1, ..., n)$.

2. The origin x_0 is unstable if Re($\lambda_i > 0$), $\exists i \in (1, ..., n)$.

3 Generalized polynomial chaos and sparse polynomial chaos expansions

3.1 Generalized polynomial chaos expansion

The generalized polynomial chaos (gPC) has been proposed by Xiu and Karniakakis [11] as a generalization of the original Wiener-Chaos expansion. The gPC establishes a separation between the stochastic components of a random function and its deterministic components. From the Wiener theory and the generalizes Cameron-Martin theorem, any second order random process $X(\xi)$ can be expanded in a convergent (in the mean square sense) polynomial function series as

$$X(\xi) \approx \sum_{\alpha \in \mathbb{N}^r} \bar{x}_\alpha \phi_\alpha(\xi), \tag{3}$$

where $\phi_\alpha(\xi)$ are orthogonal polynomial functions, $\xi(\xi_1, ..., \xi_r)$ is a vector of r independent random variables. The continuous distributions of random variables correspond to the families of orthogonal polynomials (see e.g. Table 1) [11]. Finally, $\bar{x}_\alpha$ are the polynomial chaos coefficients. In the following, $X(\xi)$ will be called the quantity of interest.

The subscript α is a multi-index, $\alpha = \{\alpha_1, ..., \alpha_r\}$ in $\mathbb{N}^r$. The length of a multi-index α is defined by

$$|\alpha| = \|\alpha\|_1 = \sum_{i=1}^{r} \alpha_i. \tag{4}$$

The corresponding index set is

[1] If the eigenvalues of the Jacobian matrix of the nonlinear system evaluated at the fixed point have non zero real parts the fixed point is hyperbolic.

Table 1: Correspondence between the families of orthogonal polynomials and random variable

Random variable ξ	The polynomial family $\phi_\alpha(\xi)$	Support
Gaussian	Hermite	(-∞,∞)
Gamma	Laguerre	[0,∞)
Beta	Jacobi	[-1,1]
Uniform	Legendre	[-1,1]

$$\mathcal{A}^{r,p} = \left\{\alpha \in \mathbb{N}^r : \|\alpha\|_1 \leq p\right\}, \tag{5}$$

with p the degree of the PC expansion.

The number of terms N_p in the gPC expansion, is shown to be dependent on the higher degree p of the polynomials $\phi_\alpha(\xi)$ and on the stochastic dimension r, i.e. the number of uncertain parameters [11]

$$N_p = card(\mathcal{A}^{r,p}) = \frac{(p+r)!}{p!r!}. \tag{6}$$

In the regression approach, the coefficients are determined through the minimization of the following least square criterion [12]

$$\varepsilon_{reg}^2 = \sum_{q=1}^{Q} \left[X\left(\xi^{(q)}\right) - \sum_{\alpha \in \mathcal{A}^{r,p}} \bar{x}_\alpha \phi_\alpha\left(\xi^{(q)}\right)\right]^2, \tag{7}$$

where $\xi^{(q)} = \left(\xi_1^{(q)}, ..., \xi_r^{(q)}\right)$, with $q = 1, ..., Q$, is the Numerical Experimental Design (NED) and $X\left(\xi^{(q)}\right)$ is the corresponding model evaluations vector. The NED $\xi^{(q)}$ with $q = 1, ..., Q$ may be built using Latin Hypercube Samples method (LHS) [13].

The coefficients are computed as follows

$$\bar{x} = \left(\phi^T(\xi^{(q)})\phi(\xi^{(q)})\right)^{-1} \phi^T(\xi^{(q)})X(\xi^{(q)}), \tag{8}$$

with $\phi(\xi^{(q)})$ the matrix defined by

$$\phi(\xi^{(q)}) = \begin{pmatrix} \phi_0(\xi^{(1)}) & \ldots & \phi_{N_p-1}(\xi^{(1)}) \\ \vdots & \ddots & \vdots \\ \phi_0(\xi^{(Q)}) & \ldots & \phi_{N_p-1}(\xi^{(Q)}) \end{pmatrix}. \tag{9}$$

3.2 Multi-element generalized polynomial chaos expansion

Let $\bar{\xi}(\bar{\xi}_1, ..., \bar{\xi}_r)$ denote a ramdom input vector where $\bar{\xi}_i$ is independent uniform random variable within the orthogonal interval $[-1, 1]^r$. Next, the space of the random input is decompose into m non-intersecting elements [14]. A local variables in each element $\zeta^k(\zeta_1^k, ..., \zeta_r^k)$ are mapped to a new independent random uniform variables $\bar{\xi}^k(\bar{\xi}_1^k, ..., \bar{\xi}_r^k)$ in $[-1, 1]^r$

$$\zeta_i^k = (b_i^k + a_i^k)/2 + \bar{\xi}_i^k(b_i^k - a_i^k)/2 \quad i = 1, ..., r;\ k = 1, ..., m \tag{10}$$

where a_i^k, b_i^k are the lower and upper bounds of the local variables ζ_i^k.

Then gPC can be used locally of the k^{th} element, the second random process corresponding to the k^{th} element is given by

$$X^k(\bar{\xi}^k) \approx \sum_{\alpha \in \mathbb{N}^r} \bar{x}^k_\alpha \phi_\alpha(\bar{\xi}^k). \tag{11}$$

3.3 Sparse polynomial chaos expansion

3.3.1 Isotropic hyperbolic index sets

One defines a hyperbolic truncation from $m-$norms, $0 < m < 1$ [9]

$$\mathcal{A}^{r,p}_m = \big\{ \alpha \in \mathbb{N}^r : \|\alpha\|_m \leq p \big\}, \tag{12}$$

where

$$\|\alpha\|_m = \left(\sum_{i=1}^{r} \alpha_i^m \right)^{1/m}. \tag{13}$$

Whatever the choice of the value of m-norms, the sequence of nested sets $\mathcal{A}^{r,p}_m$ ($p \in \mathbb{N}$) always converge to the sets $\mathbb{N}^r$. Therefore, one defines the isotropic hyperbolic polynomial chaos expansions with the index sets $\mathcal{A}^{r,p}_m$

$$X_{\mathcal{A}^{r,p}_m}(\xi) = \sum_{\alpha \in \mathcal{A}^{r,p}_m} \bar{x}_\alpha \phi_\alpha(\xi). \tag{14}$$

3.3.2 Error estimates of the polynomial chaos approximations

The strategy consists here in performing an incremental search of the significant terms. For that, it is necessary to define the error estimates of polynomial chaos approximations.

Empirical error. Consider a NED $\xi^{(q)} = \left(\xi_1^{(q)}, ..., \xi_r^{(q)}\right)$, with $q = 1, ..., Q$, and $X\left(\xi^{(q)}\right)$ is the corresponding model evaluation. $\widehat{X}_{\mathcal{A}}(\xi)$ is computed by CP approximation

$$\widehat{X}_{\mathcal{A}}(\xi) \approx \sum_{\alpha \in \mathcal{A}} \bar{x}_\alpha \phi_\alpha(\xi) \tag{15}$$

where index set $\mathcal{A}$ is a finite non empty subset of $\mathbb{N}^r$.

The generalization error is defined by mean of the squared difference between $X(\xi_i^{(q)})$ and $\widehat{X}_{\mathcal{A}}(\xi_i^{(q)})$

$$Err = \mathbb{E}\big[X(\xi_i^{(q)} - \widehat{X}_{\mathcal{A}}(\xi_i^{(q)}))^2\big]. \tag{16}$$

In practice, the generalization error may be estimated by the empirical error defined by

$$Err_{emp} = \frac{1}{Q}\sum_{q=1}^{Q}\left[\left(X(\xi^{(q)}) - \widehat{X}_{\mathcal{A}}(\xi^{(q)})\right)^2\right]. \tag{17}$$

Of common use is the related coefficient of determination R^2 which reads

$$R^2 = 1 - \frac{Err_{emp}}{\widehat{\mathbb{V}}[X]}, \tag{18}$$

where $\widehat{\mathbb{V}}[X]$ is the variance of $X\left(\xi^{(q)}\right)$

$$\widehat{\mathbb{V}}[X] = \frac{1}{Q-1}\sum_{q=1}^{Q}\left(X(\xi^{(q)}) - \bar{X}\right)^2, \qquad \bar{X} = \frac{1}{Q}\sum_{q=1}^{Q}X(\xi^{(q)}). \tag{19}$$

The empirical error underestimates the generalization error because of the overfitting phenomenon. To avoid this phenomenon, the leave-one-out error, which is a predicted residual sum of squares, is defined below [9].

Leave-one-out error. Let $\widehat{X}_{\mathcal{A}}^{(-i)}\left(\xi^{(q)}\right)$ be the chaos polynomial expansion that has been built from the NED $\left(\xi^{(1)}, ..., \xi^{(Q)}\right)\setminus\xi^{(i)}$, i.e. when removing the i-th observation $\xi^{(i)}$ from the training set $\left(\xi^{(1)}, ..., \xi^{(Q)}\right)$. The predicted residual is defined as the difference between the model evaluation at $\xi^{(i)}$ and its prediction based on $\widehat{X}_{\mathcal{A}}^{(-i)}\left(\xi^{(q)}\right)$ [9]

$$\Delta^{(i)} = X(\xi^{(q)}) - \widehat{X}_{\mathcal{A}}^{(-i)}(\xi^{(q)}). \tag{20}$$

The leave-one-out error so-called the prectited residual sum of squares is defined by

$$Err_{LOO} = \frac{1}{Q}\sum_{i=1}^{Q}\left(\Delta^{(i)}\right)^2. \tag{21}$$

It is possible to calculate analytically each predicted residual as follows [9]

$$\Delta^{(i)} = \frac{X(\xi^{(q)}) - \widehat{X}_{\mathcal{A}}(\xi^{(q)})}{1-h_i}, \tag{22}$$

here h_i is the i-th diagonal term of the matrix $\phi(\xi^{(q)})(\phi^T(\xi^{(q)})\phi(\xi^{(q)}))^{-1}\phi^T(\xi^{(q)})$. Therefore the leave-one-out error was given by

$$Err_{LOO} = \frac{1}{Q}\sum_{i=1}^{Q}\left(\frac{X(\xi_i^{(q)}) - \widehat{X}_{\mathcal{A}}(\xi_i^{(q)})}{1-h_i}\right)^2. \tag{23}$$

The equivalent determination coefficient of R^2 is denoted by S^2

$$S^2 = 1 - \frac{Err_{LOO}}{\widehat{\mathbb{V}}[X]}. \tag{24}$$

The coefficients R^2 and S^2 which are presented in this section are used in an algorithm in the following section to choose an optimal sparse polynomial chaos expansion.

3.3.3 Algorithm applied with the sparse PC expansion

An iterative procedure is presented to build the sparse PC approximation. The objective is to find the best PC coefficients [9].

The algorithm is summarized in 5 basic steps:

Step 1

- Select a numerical experiment design ($\xi^{(q)}$), e.g. a random design based on Latin Hypercube Sampling [13]. The model evaluations at the design points are gathered in vector $X(\xi_i^{(q)})$.
- Select the values of the algorithm parameters, i.e. the target accuracy S^2_{target}, the maximal PC degree p_{max}, $m-$norms of truncation strategy and the cut-off values ϵ_1, ϵ_2.

Step 2

- Initialize the algorithm: $p = 0$: the truncation index set $\mathcal{A}^0 = \{0\}$ with $\{0\}$ is the null element of $\mathbb{N}^r$.

Step 3: Training step - Enrichment of the pc basis (valid for any degree $p \in [1, ..., p_{max}]$)

⇒ **Forward step** (Addition step):

- Gather the candidate terms in a set $\mathcal{C}_m^{r,p} = \left\{\alpha \in \mathbb{N}^r : p-1 \leq \|\alpha\|_m \leq p\right\}$, add each candidate term to set $\mathcal{A}_m^{r,p-1}$ one-by-one and compute the CP expansion coefficients by regression using Eq. 8 and the associated determination coefficient R^2 in each case.
- Retain eventually those candidate terms that lead to a significant increase in the coefficient R^2, i.e. which significantly decrease the empirical error Err_{emp}, and discard the other candidate terms. Let $\mathcal{A}_m^{r,p+}$ be the final truncation set at this stage.

⇒ **Backward step** (Elimination step):

- Remove in turn each term in $\mathcal{A}_m^{r,p+}$ of degree strictly less than p. In each case, compute the PC expansion coefficients and the associated determination coefficient R^2.
- Eventually discard from $\mathcal{A}_m^{r,p+}$ those terms that lead to an insignificant decrease in R^2, i.e. a negligible increase of the empirical error Err_{emp}. Let $\mathcal{A}_m^{r,p}$ be the final truncation set.

Step 4

- Verify the conditioning of the regression information matrix. If it is poor, i.e. the size of the NED ($\xi^{(q)}$) is smaller than $2card(\mathcal{A}_m^p)$ with isotropic index sets, a enrichment of the numerical experiment design is done. In this case, the truncation set $\mathcal{A}$ is reset to $\{0\}$ and the basis enrichment procedure is restarted using nested Latin Hypercube designs [15].

Step 5: Test step

- Stop if either the leave-one-out error S^2 is less than the target S^2_{target} or the order of the PC basis is equal to p_{max}.

The algorithm applied with the isotropic index sets for building up a sparse polynomial chaos expansion is presented in figure 1.

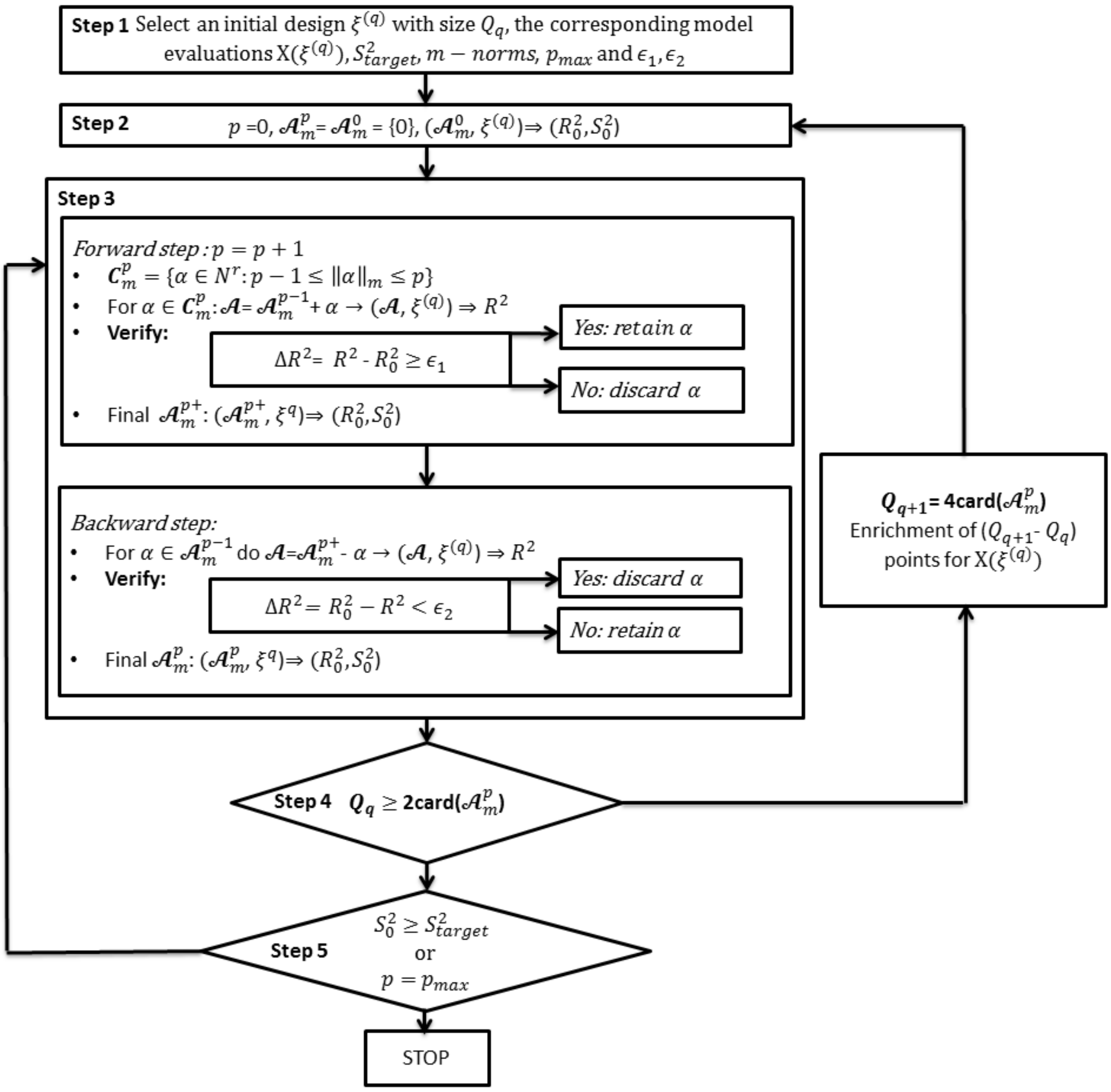

Figure 1: The algorithm applied with the Isotropic hyperbolic index sets for building up a sparse polynomial chaos expansion

4 Analytical model of the clutch system

The lumped parameter model of figure 2 that used in this study was defined by P. Wickramarachi [3] considering in addition the internal damping of the clutch disk and the pressure plate. This model with 6 degrees of freedom has been chosen because it is sufficient and efficient to study the instabilities generated by mode couplings and has been validated by experimental results. In this model, the contact

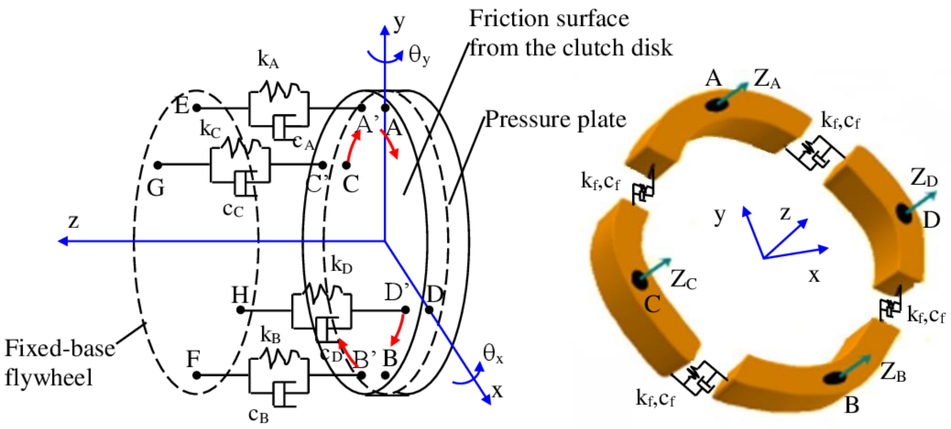

Figure 2: Analytical model of the clutch system

between the flywheel and the friction surface of the clutch system is defined at the points A', B', C'and D' by the stiffness of the cushion springs k_p which were divided into four stiffness k_A, k_B, k_C and k_D. Moreover, in order to consider the nonlinear characteristic of the clutch disk, the stiffnesses k_A, k_B which depict the first two nodal-diameter bending modes respectively multiplied and divided γ_1. Similarly, the stiffnesses k_C, k_D can be made by respectively multiplied and divided γ_2 (see Eq 30) [3]. The normal spring forces $N_{A'}$, $N_{B'}$, $N_{C'}$, $N_{D'}$ and the corresponding friction forces $F_{A'}$, $F_{B'}$, $F_{C'}$, $F_{D'}$ at the points A', B', C'and D' are given by Eq. (31) and Eq. (32).

The dampings c_A, c_B, c_C and c_D which represent the internal damping of the clutch disk are respectively placed in the same positions as the springs k_A, k_B, k_C and k_D. The points A, B, C, D are the projected points of contact on the average surface of the pressure plate. The pressure plate which was modeled as four masses, $M_p/4$ connected by the bending stiffness and damping (k_f, c_f). Points E, F, G, H are fixed points of the flywheel. The six degrees of freedom of the pressure plate are the internal rotations θ_x, θ_y around the fixed axes x, y and the translational movements Z_A, Z_B, Z_C, Z_D of points A, B, C, D which describe rigid-body rotations (wobbling modes) along the axis z.

The motion equation of the lumped parameter model of the clutch system is

$$M \cdot \ddot{U} + C \cdot \dot{U} + K \cdot U = 0 \tag{25}$$

with

$$U = \begin{bmatrix} \theta_x & \theta_y & Z_A & Z_B & Z_C & Z_D \end{bmatrix}^T, \tag{26}$$

$$M = diag\left(\begin{bmatrix} I_x & I_y & \frac{M_p}{4} & \frac{M_p}{4} & \frac{M_p}{4} & \frac{M_p}{4} \end{bmatrix}\right), \tag{27}$$

$$K = \begin{bmatrix} r^2(k_A+k_B+4k_f) & -\mu lr(k_C+k_D) & r(k_A+2k_f) & -r(k_B+2k_f) & \mu lk_C & -\mu lk_D \\ -\mu lr(k_A+k_B) & r^2(k_C+k_D+4k_f) & -\mu lk_A & \mu lk_B & r(k_C+2k_f) & -r(k_D+2k_f) \\ r(k_A+2k_f) & 0 & k_A+2k_f & 0 & -k_f & -k_f \\ -r(k_B+2k_f) & 0 & 0 & k_B+2k_f & -k_f & -k_f \\ 0 & r(k_C+2k_f) & -k_f & -k_f & k_C+2k_f & 0 \\ 0 & -r(k_D+2k_f) & -k_f & -k_f & 0 & k_D+2k_f \end{bmatrix}, \quad (28)$$

$$C = \begin{bmatrix} r^2(c_A+c_B+4c_f) & -\mu lr(c_C+c_D) & r(c_A+2c_f) & -r(c_B+2c_f) & \mu lc_C & -\mu lc_D \\ -\mu lr(c_A+c_B) & r^2(c_C+c_D+4c_f) & -\mu lc_A & \mu lc_B & r(c_C+2c_f) & -r(c_D+2c_f) \\ r(c_A+2c_f) & 0 & c_A+2c_f & 0 & -c_f & -c_f \\ -r(c_B+2c_f) & 0 & 0 & c_B+2c_f & -c_f & -c_f \\ 0 & r(c_C+2c_f) & -c_f & -c_f & c_C+2c_f & 0 \\ 0 & -r(c_D+2c_f) & -c_f & -c_f & 0 & c_D+2c_f \end{bmatrix}, \quad (29)$$

$$\begin{aligned} k_A &= \gamma_1 k_p/4; \quad k_B = k_p/(4\gamma_1); \\ k_C &= \gamma_2 k_p/4; \quad k_D = k_p/(4\gamma_2), \end{aligned} \quad (30)$$

$$\begin{aligned} N_{A'} &= k_A(Z_A - r\theta_y) \\ N_{B'} &= k_B(Z_B + r\theta_y) \\ N_{C'} &= k_C(Z_C + r\theta_x) \\ N_{D'} &= k_D(Z_D - r\theta_x), \end{aligned} \quad (31)$$

$$\begin{aligned} F_{A'} &= \mu N_{A'} \\ F_{B'} &= \mu N_{B'} \\ F_{C'} &= \mu N_{C'} \\ F_{D'} &= \mu N_{D'}, \end{aligned} \quad (32)$$

where $r = (r_1 + r_2)/2$ with r_1 and r_2 are the minimum and maximum sliding radi, μ is the friction coefficient which is assumed constant because of the the high slip speed (<700 rev/min) and l is the thickness of the pressure plate. The nominal values of the parameters are: k_p = 16 MN/m; k_f = 7 MN/m; γ_1 = 0.9; γ_2 = 0.8; r_1 = 75 mm; r_2 = 120 mm; l = 12.5 mm; $c_A = c_B = c_C = c_D = 4Nm^{-1}s^{-1}$; $c_f = 0.1Nm^{-1}s^{-1}$ [3].

5 Application and results

The objective of this section is to study the stability of a clutch system with uncertain parameters. These studies are done in the lumped parameter presented in section 4. There are eight parameters which can be uncertain: $\mu, k_p, k_f, \gamma_1, \gamma_2, r_1, r_2$ and l. The uncertain parameters are considered as independent random and uniform in the intervals [0.95*V_n, 1.05*V_n] with V_n the corresponding nominal values; μ is independent random and uniform in the interval [0, 0.5]. The nominal values of the parameters have been given in section 4.

5.1 Reference study: Monte Carlo on the initial model

In this section, the stability of the clutch system is investigated using the Monte Carlo on the initial model (MCIM). 10,000 eigenvalues $\lambda_i (i = 1, ..., n)$ of the Jacobian matrix (Eq. (2)) of the clutch system are calculated directly from a set of 10,000 independent random samples.

In the eigenvalues of the clutch system, there are four non-zeros eigenvalues (modes 1, 2, 3 and 4 with the complex conjugates). Modes 3 et 4 are always decoupled and stable and the coalescence phenomenon can occur between the modes 1 and 2. Therefore, the stability of the clutch system only depends on these two modes.

Figure 3 shows the real and imaginary parts of mode 1 (blue) and mode 2 (red) and highlights the coalescence phenomenon of eigenvalues described hereafter. From $\mu = 0$, the imaginary parts (frequencies) of these two modes are separated and tend to come closer when the coefficient of friction increases. As for the real parts, they are both negative, i.e. the system is stable. Then, the imaginary parts of the two modes overlap, it is the so-called mode coalescence phenomenon. Beyond this point, imaginary parts are almost equal. From the Hopf bifurcation point, if μ increases, the real part of one mode is positive and the system becomes therefore unstable.

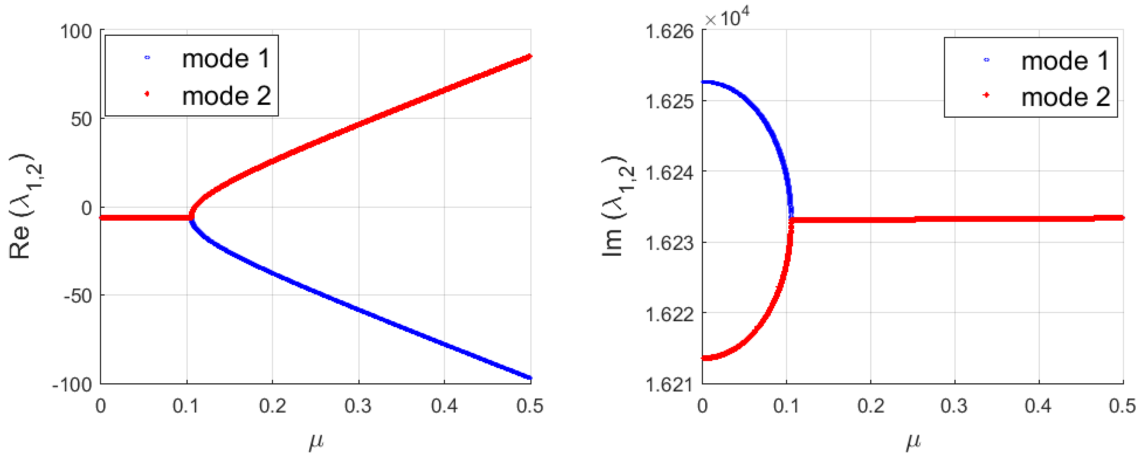

Figure 3: (a) Real part and (b) imaginary parts of the eigenvalues λ_1 and λ_2 for the modes 1 and 2

5.2 Sparse PC method for the stability analysis of a clutch system

In this section, the distribution of the eigenvalues of the clutch system is rebuild by coupling the MC method with the polynomial chaos expansion methods presented in section 3.

Table 2 presents the comparison between the results obtained with gPC, ME-gPC, Sparse PC and MCIM. The results using gPC and MC-gPC were computed by Trinh et al. [8]. The results show that gPC can be used in the stability analysis of the clutch system with a small number of uncertain parameters r (up to 5). With a high value of r, the number of Direct Computations (DC) of the initial system needed to determine the PC coefficients became too high and the relative errors of the proportion stability between gPC and MCIM are higher than 20% from $r = 6$. Te use of the ME-gPC is effective for $r \leq 7$. However, with $r = 5 \rightarrow 7$, the numbers of DC of the complete system are rather high (> 3000), that means the computational costs are expensive. With $r = 8$, the number of DC to determine the coefficients of the PC expansion exceed the limited number (10,000) for both methods.

With Sparse PC, authors compute five sets of NED to analyze the influence of the choice of the NED on the Sparse PC method. A target accuracy $S^2_{target} = 0.999$ and a maximal PC degree $p_{max} = 15$ are used. In table 2, the results of each of the five NED are not all depicted, only the minimal and maximal values are shown in square brackets. The results show that the use of this method reduce remarkably the number of DC compare with gPC and ME-gPC, specially for $r \geq 5$. Moreover, the method ensures a high accuracy compared with gPC. The relative errors of the proportion of stability using Sparse PC are lower than 5% for $r = 3 \rightarrow 5$ than 7% for $r = 6 \rightarrow 8$

For $r = 8$, gPC and ME-gPC are not effective because the number of DC exceeds the limited number whereas the number of DC using Sparse PC lower than 3000. It means a saving of 70% of the computational cost with a high accuracy ($< 7\%$).

Figure 4 shows the eigenvalue real parts of modes 1 and 2 using MCIM (blue) and using Sparse PC (red) (for $r = 3, 4, 5, 6, 7, 8$). The results computed with Sparse PC are close to the results obtained via MCIM. The Hopf bifurcation points with Sparse and MCIM are also close. Therefore, Sparse PC ensures a high accuracy of the results.

6 Conclusion

The computational cost of gPC and ME-gPC in studying the stability analysis of a clutch system can become prohibitive with a large number of uncertain parameters ($r \geq 5$). While using Sparse PC with Isotropic hyperbolic index sets allows a remarkable reduction of the computational cost by ensuring a high accuracy compared with these common PC expansion. With number of uncertain parameters r = 8, gPC and ME-gPC were not feasible because of the limited number of DC. Whereas using Sparse PC isotropic hyperbolic index sets reduce even more than 70% of the computational cost with a high accuracy (the relative errors of the proportion stability are lower than 7% with Sparse PC anisotropic hyperbolic index sets.

In conclusion, Sparse polynomial chaos is efficient for high number of uncertain parameters ($r = 3 \rightarrow 8$ in this study). It reduces remarkably the computational cost by ensuring a high accuracy compared with the common polynomial chaos expansion.

References

[1] B. Hervé, J.-J. Sinou, H. Mahé, and L. Jézéquel. Extension of the destabilization paradox to limit cycle amplitudes for a nonlinear self-excited system subject to gyroscopic and circulatory actions. *Journal of Sound and Vibration*, 323(3):944 – 973, 2009.

[2] F. Van De Velde and P. De Baets. A new approach of stick-slip based on quasi-harmonic tangential oscillations. *Wear*, 216(1):15 – 26, 1998.

[3] Pierre Wickramarachi, Rajendra Singh, and George Bailey. Analysis of friction-induced vibration leading to "eek" noise in a dry friction clutch. *Noise control engineering journal*, 53(4):138–144, 2005.

Table 2: Comparison of the results between gPC, ME-pPC, Sparse PC and MCIM

Number of uncertain parameters (r)	Title	Order (p)	Number of DC of complete system	Relative mean error between sparse PC /gPC/ME-gPC and MCIM (%)	Relative variance error between sparse PC /gPC/ME-gPC and MCIM (%)	Relative error of the proportion stability between sparse PC gPC/ME-gPC and MCIM (%)
-	Direct Computation (DC)	-	10000	-	-	-
1	gPC	6	7	0.074	0.35	6.45
2	gPC	6	49	0.074	0.35	0.18
	ME-gPC	2	126	0.05	0.21	0.18
			234	0.047	0.020	0.18
3	gPC	6	343	0.74	0.35	6.45
	ME-gPC	2	378	0.05	0.02	0.227
			756	0.047	0.198	0.227
	Sparse PC	[8;11]	[195;235]	[1.25;1.31]	[0.58;0.96]	[0.05;2.65]
4	gPC	6	2401	0.25	0.78	7.54
	ME-gPC	2	648	0.027	0.19	1.22
			972	1.5e-4	0.003	0.59
	Sparse PC	[10;13]	[175;482]	[0.95;1.33]	[0.06;2.04]	[1.20;4.16]
5	gPC	5	7776	0.049	0.61	8.04
	ME-gPC	2	3402	0.008	0.31	1.68
			5832	4.7e-4	0.28	0.59
	Sparse PC	[15]	[922;1187]	[1.17;1.39]	[0.13;1.00]	[4.39;4.96]
6	gPC	3	4096	0.72	4.13	26.85
	ME-gPC	2	4374	0.01	0.33	1.04
			5103	0.018	0.35	1.04
	Sparse PC	[15]	[933;1399]	[1.07;1.47]	[0.17;1.10]	[3.94;6.82]
7	gPC	2	2187	0.71	1.08	43.84
	ME-gPC	2	8748	0.01	0.23	2.45
			8748	0.01	0.23	2.45
	Sparse PC	[15]	[1045 ;1708]	[0.96;1.27]	[0.31;1.10]	[3.06 ;6.89]
8	gPC	2	6561	0.70	1.10	43.88
	Sparse PC	[15]	[1706 ;2117]	[1.18;1.32]	[0.13;0.92]	[3.15 ;5.85]

[4] Guillaume Fritz, Jean-Jacques Sinou, Jean-Marc Duffal, and Louis Jézéquel. Investigation of the relationship between damping and mode-coupling patterns in case of brake squeal. *Journal of Sound and Vibration*, 307(3):591 – 609, 2007.

[5] D. Centea, H. Rahnejat, and M.T. Menday. Non-linear multi-body dynamic analysis for the study of clutch torsional vibrations (judder). *Applied Mathematical Modelling*, 25(3):177 – 192, 2001.

[6] Lyes Nechak, Sébastien Berger, and Evelyne Aubry. A polynomial chaos approach to the robust analysis of the dynamic behaviour of friction systems. *European Journal of Mechanics - A/Solids*, 30(4):594 – 607, 2011.

[7] E. Sarrouy, O. Dessombz, and J.-J. Sinou. Piecewise polynomial chaos expansion with an application to brake squeal of a linear brake system. *Journal of Sound and Vibration*, 332(3):577 – 594, 2013.

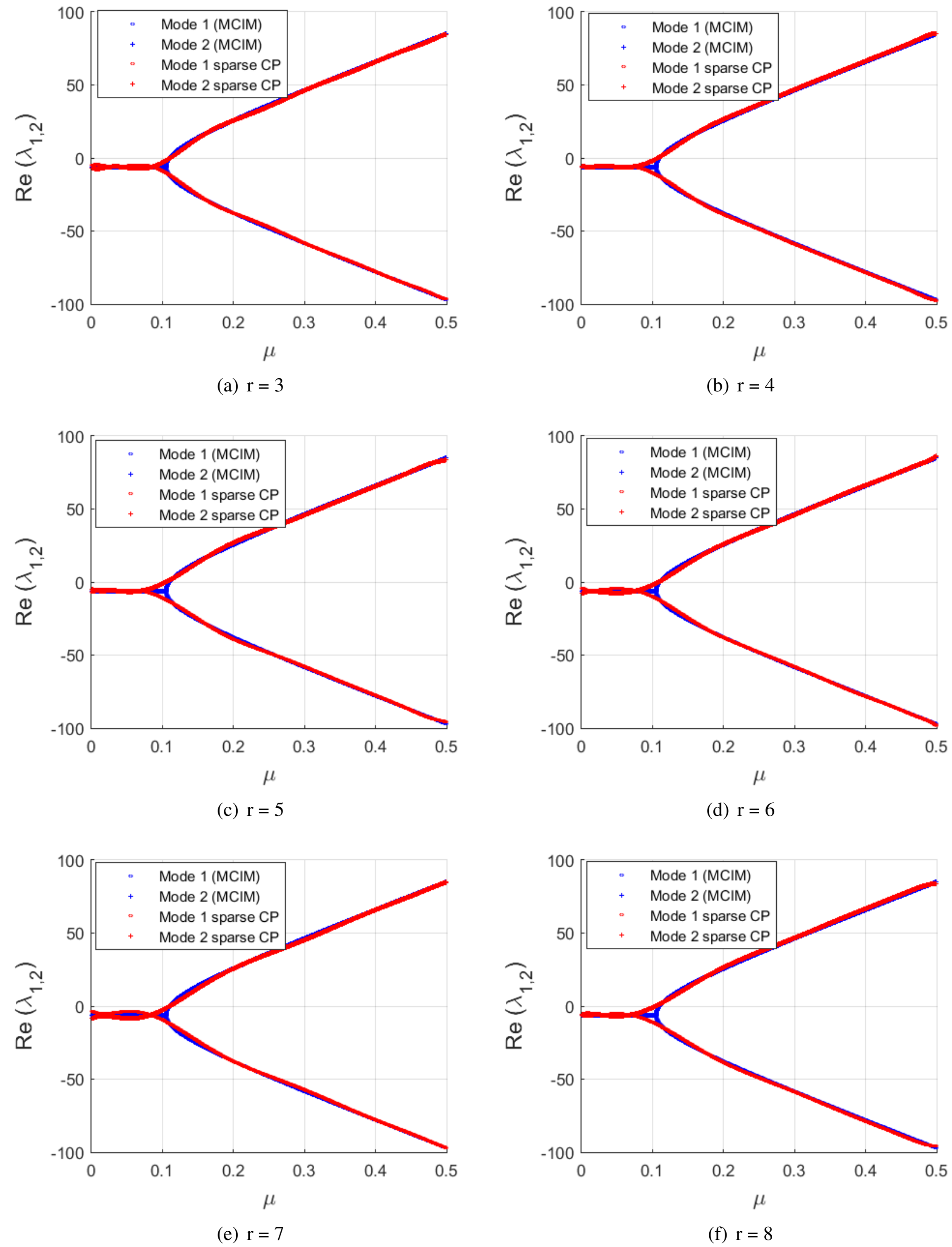

Figure 4: Real parts of mode 1 and 2 using MCIM and Sparse PC

[8] Trinh Minh Hoang, Berger Sébastien, and Aubry Evelyne. Stability analysis of a clutch system with multi-element generalized polynomial chaos. *Mechanics & Industry*, 17(2):205, 2016.

[9] Géraud Blatman and Bruno Sudret. Anisotropic parcimonious polynomial chaos expansions based on the sparsity-of-effects principle. In *Proc ICOSSAR'09, International Conference in Structural*

Safety and Relability, 2009.

[10] Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media, 2003.

[11] Dongbin Xiu and George Em Karniadakis. The wiener–askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2):619–644, 2002.

[12] Marc Berveiller, Bruno Sudret, and Maurice Lemaire. Stochastic finite element: a non intrusive approach by regression. *European Journal of Computational Mechanics/Revue Européenne de Mécanique Numérique*, 15(1-3):81–92, 2006.

[13] M. D. McKay, R. J. Beckman, and W. J. Conover. A Comparison of Three Methods for Selecting Values of Input Variables in the Analysis of Output from a Computer Code. *Technometrics*, 21(2):239–245, 1979.

[14] Xiaoliang Wan and George Em Karniadakis. An adaptive multi-element generalized polynomial chaos method for stochastic differential equations. *Journal of Computational Physics*, 209(2):617 – 642, 2005.

[15] G. Gary Wang. Adaptive Response Surface Method Using Inherited Latin Hypercube Design Points. *Journal of Mechanical Design*, 125(2):210–220, June 2003.