Some Bayesian insights for statistical tolerance analysis

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Abstract:

Functionality of assembled products mostly rely on the ability of the manufacturer to produce under some quality requirements. Parts which do not meet these requirements represent a manufacturing waste which can be at the origin of substantial losses in terms of money and credibility. Quality control and defect detection are two keypoints of predictive process management. At the design stage, a statistical tolerance analysis can be performed to predict the process quality. This imply to estimate a so-called defect probability which quantifies the probability that the final assembly does not meet functional requirements. In general, this quantity depends on a number of process specifications (tolerances, capability levels) set a priori by the manufacturer, but also on the monitoring of the process itself since the process parameters (mean shift value and standard deviation) vary statistically for different batches. In this paper, we give an alternative point of view on an existing method, namely the Advanced Probability-based Tolerance Analysis of products (APTA), proposed in literature to estimate the defect probability. This method, originally relying on a double-loop sampling strategy, is revisited within the Bayesian framework, and an augmented approach is proposed to estimate the defect probability in a more efficient way. The efficiency of the augmented approach for solving tolerancing problems with APTA is illustrated on a linear reference test-case.

Keywords: APTA / tolerance analysis / defect probability / reliability / Bayesian approach

1 Introduction

Variability in the manufacturing process is an inherent drawback of mass production [1]. It means that some manufactured parts may have out-of-tolerance dimensions due to various factors such as varying material properties, tool wear or human errors. In order to improve quality and cost effectiveness of a process, it is necessary to take into account these sources of uncertainty. Indeed, the future system performance may also directly depend on the quality of the manufactured components and assembly, which implies the way the process has been controled and adjusted.

In the literature, uncertainty is usually decomposed into *aleatory* uncertainty, supposed to represent natural variability (considered as irreducible in a specific context), and *epistemic* uncertainty, ensuing from a lack of knowledge or mathematical simplifications and which can be reduced by adding more information or increasing the model fidelity. This conceptual dichotomy is problem-dependent, as pointed out in [2], and should be interpreted as a practical guideline to discern between uncertainties that can be reduced with a given reasonable effort (e.g. by gathering more data, running more simulations or refining the model) and those which cannot. In this paper, the problem under consideration combines both types of uncertainty since the part dimensions vary due to aleatory uncertainty (e.g., variability in material properties) and epistemic uncertainty (e.g., tool wear). Moreover, the process parameters themselves (roughly speaking, the mean and the standard deviation of the batches sampled for statistical process control) are effected by various sources which can be either seen as aleatory or epistemic (e.g., *statistical* uncertainty arising from measuring procedures [3] or possible human errors from the machine tool operator). For these reasons, it is extremely difficult to detect and qualify the role of these uncertainty sources and to determine whether they can be reduced or not. These considerations depend on the ability of the manufacturer to improve its quality control policy and management. In this paper, we will only focus on the fact that we are confronted to a *bi-level uncertainty* stemming from both part dimensions and process parameters. Finally, the process quality can be quantitatively measured by estimating a so-called *defect probability* P_D, measuring the number of defective parts per million with respect to the uncertainties affecting the process.

In this paper, we consider a fully probabilistic framework by assuming that uncertain quantities are modeled by random variables. Our aim is to illustrate how the *Advanced Probability-based Tolerance Analysis of products* (APTA) method [4], developed to estimate the defect probability by taking jointly into account both uncertainties arising from the basic random variables (i.e. the variability of the dimensions of manufactured parts) and from the process parameters (i.e. the mean shift value and the standard deviation of a given manufactured batch of parts) can be numerically enhanced using a Bayesian predictive approach [5], namely the *augmented approach*. This Bayesian approach enables to solve the numerical probability estimation in a more efficient way in terms of number of calls to the tolerance chain function, which can be possibly a computer model of the assembly or another costly-to-evaluate function.

The paper is organized as follows. Section 2 recalls basic principles of statistical tolerance analysis and gives an overview of APTA. Section 3 introduces the augmented approach and details how it can be relevant for enhancing the sampling strategy inherent of APTA. Section 4 aims at demonstrating the efficiency of the augmented approach coupled to APTA for solving a two-part assembly test-case. Finally, Section 5 draws some conclusions and presents future works.

2 Statistical tolerance analysis and APTA

2.1 General formulation for statistical tolerance analysis

To ensure some system performances, manufacturers have first to prevent misassemblies by reaching and maintaining quality levels in terms of manufacturing process. Tolerances, often obtained from drawing dimensions, define admissible variations in the geometry and positioning of parts or subsystems [6]. The statistical treatment of these tolerances comes from the fact that manufacturing processes are affected by uncertainties (e.g., material properties variability and tool wear). Thus, for a given production batch of N samples (i.e. N manufactured parts of dimension X), one can perform a statistical treatment on these data and show that part dimension X follows a given probability distribution. This general behavior is illustrated in Figure 1.

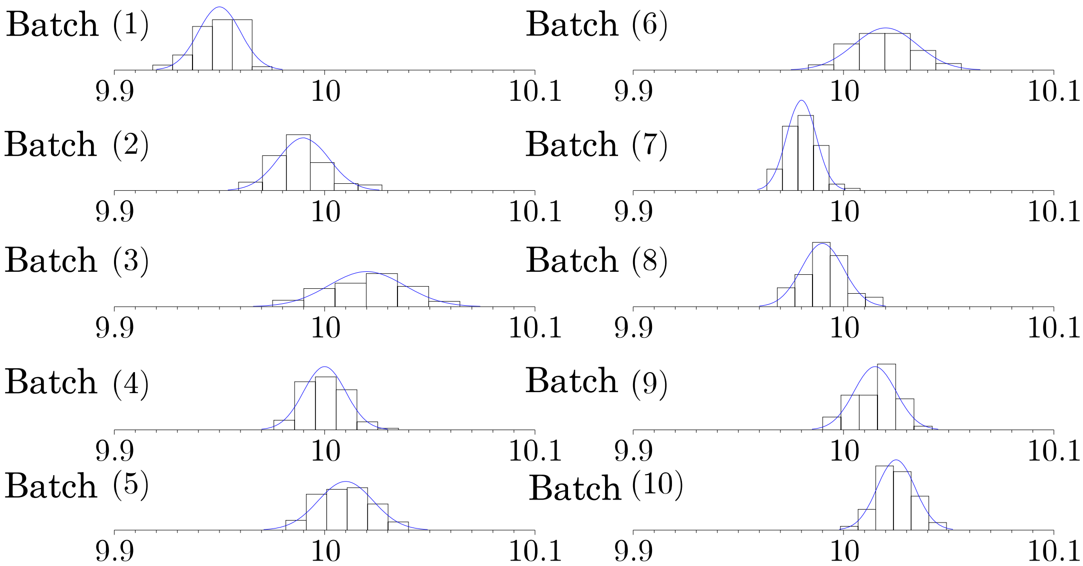


Figure 1 – Illustration of statistical variability of process parameters for 10 production batches of a part, whose nominal dimension value is $T = 10$ and tolerance interval is $t = 0.2$ (adapted from [7]).

The global functionality of the assembly can be expressed by the tolerance chain function:

$$Y = f(\mathbf{X}) \tag{1}$$

where $\mathbf{X} = (X_1, X_2, \ldots, X_d)^\top$ is the vector of d part dimensions modeled by random variables and Y is the scalar assembly response, which is also a random variable. The *tolerance chain function* $f(\cdot)$ can be either linear or nonlinear, cheap or costly-to-evaluate, explicit or not. In the latter case, the function is evaluated through a numerical simulation, such as the resolution of a finite element problem or can be obtained by a computer-aided design model. The idea is to make sure that the previous quantity of interest (QoI) Y lies in a functional interval such that:

$$Y \in [\mathrm{LSL}_Y, \mathrm{USL}_Y] \tag{2}$$

where LSL_Y and USL_Y are functional bounds (respectively the *Lower* and *Upper Specification Limits* of Y). From the manufacturer point of view, finding the good tradeoff between quality and manufacturing cost is always a critical issue. Production costs are directly impacted by how tight the required tolerances are, while quality is driven by customer requirements in terms of allowable *defect probability*, expressed in parts per million (ppm). This quantity can be defined as follows:

$$P_D = \mathbb{P}\Big[Y \notin [\mathrm{LSL}_Y, \mathrm{USL}_Y]\Big]. \tag{3}$$

Estimating this defect probability can be challenging since only a few parts per million may be defective. Here, the usefulness of the reliability framework and its set of methods to practically compute this probability is highlighted. A number of authors developed various approaches based on reliability methods for this purpose [4, 8, 9, 10, 11]. The evaluation of the defect probability depends on the complexity of the tolerance chain function $f(\cdot)$. Considering Eq. (2), one can introduce two limit-state functions, $g_1(\cdot)$ and $g_2(\cdot)$, which characterize the two different functional conditions the assembly response Y is subjected to:

$$g_1(\mathbf{X}) = f(\mathbf{X}) - \mathrm{LSL}_Y \tag{4a}$$

$$g_2(\mathbf{X}) = \mathrm{USL}_Y - f(\mathbf{X}). \tag{4b}$$

Thus, estimating P_D is similar to solving a system reliability problem for a series system modeled by the two previous limit-state functions. The probability can be expressed as follows:

$$P_D = \mathbb{P}\left[\cup_{j=1}^{2}\{g_j(\mathbf{X}) \le 0\}\right] \tag{5a}$$

$$= \mathbb{P}[g_1(\mathbf{X}) \le 0\}] + \mathbb{P}[g_2(\mathbf{X}) \le 0\}] - \mathbb{P}\left[\cap_{j=1}^{2}\{g_j(\mathbf{X}) \le 0\}\right] \tag{5b}$$

$$\le \mathbb{P}[g_1(\mathbf{X}) \le 0\}] + \mathbb{P}[g_2(\mathbf{X}) \le 0\}]. \tag{5c}$$

The last inequality in Eq. (5c) becomes an equality if the two hyperplanes associated with $g_1(\cdot)$ and $g_2(\cdot)$ result anticorrelated [12], which is equivalent to a null intersection. Such a problem is often encountered in the field of tolerance analysis, due to classical formulations of tolerance chain functions and functional bounds. For other hyperplane configurations, the interested reader may refer to [10, 11].

In mass production, each dimension X_i is a random variable, often assumed to follow a Gaussian distribution [6]. For the sake of simplicity, the variables $\mathrm{X}_i, i \in [\![1,d]\!]$ are considered to be independent. In the following, we will stick to these two assumptions since they are the most commonly encountered in literature and in many daily life cases. Other assumptions such as different types of distribution and possible correlation between part dimensions can be found in [13] but are not within the scope of the present paper. Without any loss of generality, one can assume that X_i is defined by a target value, T_i, and a tolerance interval, $t_i = \mathrm{USL}_i - \mathrm{LSL}_i$, as shown in Figure 2a. However, this Gaussian centered model implicitly assumes a quasi-perfect production process. In reality, the statistical behavior of the manufacturing process is controlled by two capability indices [14], C_{pi} and C_{pki}, respectively defined as:

$$C_{pi} = \frac{t_i}{6\sigma_i}, \quad C_{pki} = \frac{t_i/2 - |\delta_i|}{3\sigma_i} = \min\left(\frac{\mu_i - \mathrm{LSL}_i}{3\sigma_i}, \frac{\mathrm{USL}_i - \mu_i}{3\sigma_i}\right) \tag{6}$$

where σ_i and δ_i are the statistical parameters (respectively, the standard deviation and the mean shift $\delta_i = \mu_i - T_i$, with μ_i the mean of the distribution of dimension X_i) as represented in Figure 2b. In statistical process control, a production batch is considered to be admissible if the two capability indices meet the imposed requirements, i.e. $C_{pi} \ge C_{pi}^{(r)}$ and $C_{pki} \ge C_{pki}^{(r)}$. These requirements are chosen by the manufacturer in compliance with those of the customers. Thus, using the equations characterizing the capability indices, one can define a so-called *conformity domain* V_D which can be represented in two-dimensional diagram as shown in Figure 2c. To understand this diagram, one first needs to recall that the conformity domain is triangular due to the bounding caused by the capabilities in Eq. (6). Moreover, the grey area at the bottom of the diagram represents the fact that the lowest value for σ_i is $\sigma_i^{(\min)} \neq 0$ since such a value is unreachable in practice. Thus, the standard deviation σ_i belongs to an interval $\left[\sigma_i^{(\min)}, \sigma_i^{(\max)}\right]$, such that:

$$\sigma_i^{(\min)} = \frac{t_i}{6C_{pi}^{(\max)}}, \quad \sigma_i^{(\max)} = \frac{t_i}{6C_{pi}^{(r)}} \tag{7}$$

where $C_{pi}^{(\max)}$ represents the maximum capability of the variability domain. In the following, either the capability or the standard deviation will be considered as a random variable due to process variability since they are linked by Eq. (6). As a consequence, the mean shift δ_i also varies in an interval

$\left[-\delta_i^{(\max)}, \delta_i^{(\max)}\right]$ where:

$$\delta_i^{(\max)} = \frac{t_i}{2}\left(1 - \frac{C_{pki}^{(r)}}{C_{pi}^{(\max)}}\right). \tag{8}$$

The idea for the manufacturer is to estimate the defect probability associated to these process specifications and try to meet the customer's expectations regarding the number of defective parts (a customer may also want to pay for a high quality process ensuring a very small amount of defective ppm). However, a key issue remains the way we take into account in the probability estimation the fluctuations (due to some variability in material properties, tool wear or small changes in the process) affecting $(\boldsymbol{\delta}, \boldsymbol{\sigma})$ over time, with $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)^\top$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d)^\top$. Among existing approaches in literature, the *Advanced Probability-based Tolerance Analysis of products* (APTA) method relies on the concept of *dynamic shifted distribution* [15] and allows to handle two uncertainty levels: the first one consisting of basic random variables gathered in $\mathbf{X}$ (representing the variability affecting part dimensions) and the second one at the level of the process parameters $(\boldsymbol{\delta}, \boldsymbol{\sigma})$ which are themselves uncertain. To remain coherent, we will now use the notation $(\boldsymbol{\Delta}, \boldsymbol{\Sigma})$ since they are random vectors. In the next section, a brief summary of the method is provided following the original works in [4, 7].

2.2 Brief overview of the Advanced Probability-based Tolerance Analysis of products (APTA)

APTA, as originally formulated in [4, 7], mainly relies on treating the problem of tolerance analysis in a Bayesian framework, assuming that each part dimension follows some Gaussian distribution such that $\mathrm{X}_i \sim \mathcal{N}(M_i, \Sigma_i)$ (with M_i representing the mean value which also a random variable due to the relation $M_i = \Delta_i + T_i$ as shown in Figure 2b). All the X_i variables are assumed to be independent and can be gathered in the random vector $\mathbf{X}$ of joint probability density function (pdf) $f_{\mathbf{X}}(\cdot; \boldsymbol{\delta}, \boldsymbol{\sigma}) : \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d \to \mathbb{R}_+$. Moreover, for the sake of simplicity, all the distribution parameters couples (δ_i, σ_i) are supposed to be independent and $h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}(\cdot,\cdot)$ is the joint pdf defined over the conformity domain V_D. In the two previous expressions of the pdfs, one needs to pay attention to the following notations which are intensively used in this article: a comma symbol refers to joint variables while a semi-colon refers to conditional variables. Thus, the defect probability can be expressed in the APTA framework as follows:

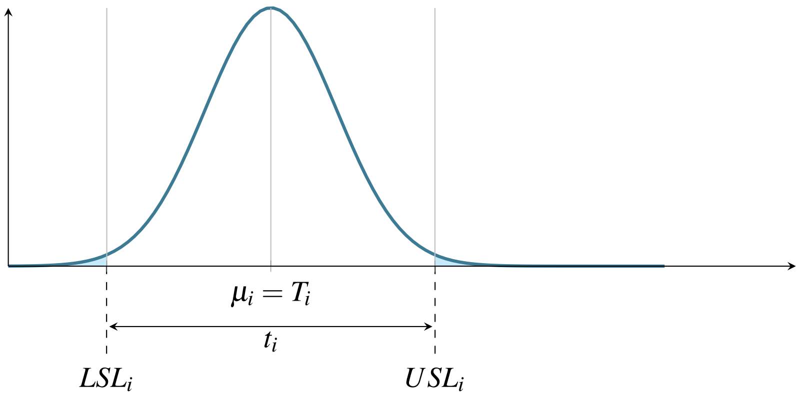
$$P_D = \mathbb{E}_{h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}}\left[P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\Delta}, \boldsymbol{\Sigma})\right] \tag{9a}$$

$$= \int_{V_D} P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}} h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}(\boldsymbol{\delta}, \boldsymbol{\sigma}) \,\mathrm{d}\boldsymbol{\delta}\, \mathrm{d}\boldsymbol{\sigma} \tag{9b}$$

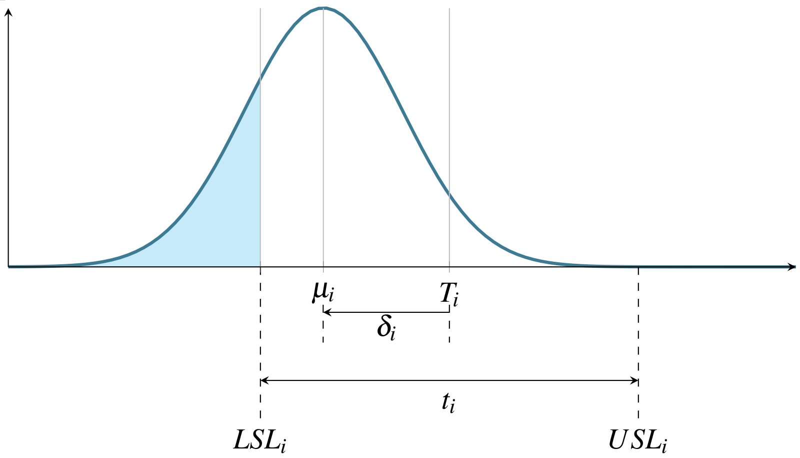
$$= \int_{V_D} \left(\int_{\cup_{j=1}^{2}\{g_j(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}; \boldsymbol{\delta}, \boldsymbol{\sigma}) \mathrm{d}\mathbf{x} \right) h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}(\boldsymbol{\delta}, \boldsymbol{\sigma}) \,\mathrm{d}\boldsymbol{\delta}\, \mathrm{d}\boldsymbol{\sigma} \tag{9c}$$

$$= \int_{V_D} \left(\int_{\cup_{j=1}^{2}\{g_j(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}; \boldsymbol{\delta}, \boldsymbol{\sigma}) \mathrm{d}\mathbf{x} \right) \prod_{i=1}^{d} h_{\Delta_i,\Sigma_i}(\delta_i, \sigma_i) \,\mathrm{d}\delta_i\, \mathrm{d}\sigma_i \tag{9d}$$

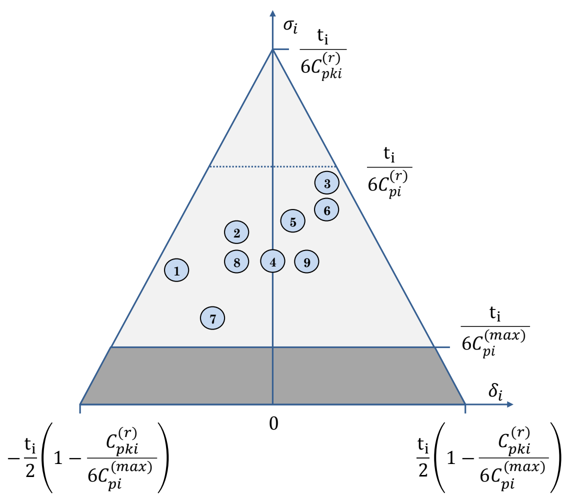
where $P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\delta}, \boldsymbol{\sigma})$ is a conditional defect probability evaluated for realizations of the pair $(\boldsymbol{\Delta}, \boldsymbol{\Sigma})$. If the function $f(\cdot)$ is linear (and the X_i are Gaussian variables), the conditional defect probability can be



(a) Illustration of a centered Gaussian manufacturing process.



(b) Illustration of a shifted Gaussian manufacturing process.



(c) Illustration of production batches mentioned in Figure 1 lying in a conformity domain (δ_i, σ_i) for a given part dimension X_i (adapted from [7]).

Figure 2 – Two Gaussian models for manufacturing processes (the blue areas represent the non-conformity area which has to be estimated by the defect probability). Below, a conformity domain as a function of the statistical process parameters and the capacity levels (the grey area represents the non-reachable domain).

computed using the following analytical formula:

$$P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\delta},\boldsymbol{\sigma}) = \Phi\left(-\frac{\mu_Y - \mathrm{LSL}_Y}{\sigma_Y}\right) + \Phi\left(-\frac{\mathrm{USL}_Y - \mu_Y}{\sigma_Y}\right) \tag{10}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard Gaussian distribution, μ_Y and σ_Y are respectively the mean and standard deviation of the output Y. Eq. (10) corresponds to the exact case where the *First-Order Reliability Method* (FORM) is licit [9, 12]. As previsouly discussed, since we are confronted to a series system reliability analysis, another FORM-based method with a search algorithm for multiple design points, known as *Multi*-FORM [16, 17], can be used in this case. If $f(\cdot)$ is nonlinear, this expression is no longer available, and FORM cannot be applied anymore since the approximation of each limit-state function by a linear hyper-plane is not a valid assumption. In this case, one needs to use another approximation method (mostly the *Second-Order Reliability Method* (SORM) [12]) or any dedicated simulation method (e.g., *Crude Monte Carlo* (CMC), *Importance Sampling* (IS), *Directional Sampling* (DS), *Line Sampling* (LS) or *Subset Sampling* (SS)). Detailed descriptions and algorithms of these sampling methods can be found in [18].

2.3 Discussion about some crucial steps of APTA

In their original paper [4], the authors explicitly use an outer Monte Carlo loop to sample over the V_D space some realizations of $(\boldsymbol{\Delta},\boldsymbol{\Sigma})$, and then evaluate, for each pair, the conditional defect probability $P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\delta},\boldsymbol{\sigma})$ using a FORM computation.

Thus, one can define three major steps in APTA which have to be discussed within the scope of this paper, i.e. from a Bayesian point of view:

- firstly, one needs to define a prior joint pdf $h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}(\cdot,\cdot)$ for process parameters $(\boldsymbol{\Delta},\boldsymbol{\Sigma})$ based on the available data (provided by a procedure of statistical process control [1]) or following some expert judgment;
- secondly, one needs to define a sampling strategy over the conformity domain V_D so as to sample a number of realizations of $(\boldsymbol{\Delta},\boldsymbol{\Sigma})$. Up to now, this step is achieved by CMC;
- thirdly, one needs to evaluate the conditional defect probability $P_{D|\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\delta},\boldsymbol{\sigma})$ for each sampled pair.

The first point is problem-dependent since it concerns a priori modeling choices, while the two other points derive from the numerical strategy and drive the computational cost and efficiency. By using a Bayesian predictive approach (named *augmented approach*), it is possible to merge the last two points and thus reducing the computational cost without loss of accuracy. Moreover, in the case where $f(\cdot)$ is a nonlinear tolerance chain function, the use of FORM to estimate the conditional defect probability could be improper and lead to erroneous results. Depending on the strength of the nonlinearity, only simulation methods could be available and licit to evaluate such a probability. Finally, the rareness of the probability could lead to dramatically increase the number of simulations. Consequently, the augmented approach can be a relevant alternative to reduce simulation cost.

3 Estimating the defect probability by an augmented approach

Dealing with a bi-level uncertainty is a problem in the field of reliability. As illustrated in some early papers [19, 20, 21, 22], taking statistical uncertainty affecting probability distribution parameters into account is of prime importance in terms of safety assessment. In [2, 23], the authors adress this topic considering parameter uncertainty affecting probability distributions within the use of FORM. However, this issue can be adressed in a more global way by considering a Bayesian framework and adopting a certain view to numerically solve the problem [24].

In the context of reliability assessment, let us consider a problem expressed in terms of a number of basic variables gathered in a d-dimensional random vector $\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta}) : \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d \to \mathbb{R}_+$. Adopting a Bayesian point of view, one can assume that the uncertainty affecting probability distribution parameters can be modeled using a prior distribution $h_{\boldsymbol{\Theta}}(\cdot;\boldsymbol{\xi})$ such that $\boldsymbol{\Theta} \sim h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\xi}) : \mathcal{D}_{\boldsymbol{\Theta}} \subseteq \mathbb{R}^k \to \mathbb{R}_+$, where $\boldsymbol{\xi}$ represents some deterministic hyper-parameters (i.e. moments or bounds) characterizing the a priori choice made following limited information or some expert judgment.

Then, performing a reliability analysis at a given value $\boldsymbol{\theta}$, realization of $\boldsymbol{\Theta}$ leads to what we call a conditional failure probability:

$$P_{\mathrm{f}}(\boldsymbol{\theta}) = \mathbb{P}\left[g(\mathbf{X}) \leq 0\right] = \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{X}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta}) \mathrm{d}\mathbf{x} \tag{11}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is the limit-state function which characterizes the behavior of the system (failure if $g(\mathbf{x}) \leq 0$, safety if $g(\mathbf{x}) > 0$) and $\mathcal{F}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{X}} : g(\mathbf{x}) \leq 0\}$ is the failure domain. Thus, following the Bayesian framework adopted in [23], one obtains the *predictive failure probability* which takes into account both uncertainties from basic variables and distribution parameters:

$$\widetilde{P}_{\mathrm{f}} = \mathbb{E}_{h_{\boldsymbol{\Theta}}}\left[P_{\mathrm{f}}(\boldsymbol{\Theta})\right] \tag{12a}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} P_{\mathrm{f}}(\boldsymbol{\theta}) h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta} \tag{12b}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} \left(\int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta}) \mathrm{d}\mathbf{x} \right) h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta}. \tag{12c}$$

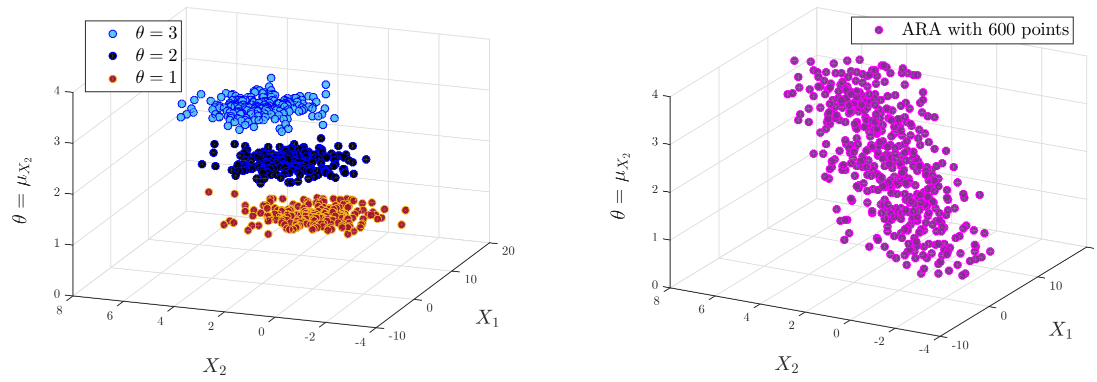
At this point, Table 1 aims at collecting elements from both types of analyzes (tolerance vs. reliability) and comparing them to make the similarities between these two fields clearer and more explicit.

Table 1 – Analogy between tolerance analysis and reliability under parameter uncertainty frameworks.

Framework	**QoI**	**Conditional QoI**	**Level 1 uncertainty** Basic variables	**Level 2 uncertainty** Distribution parameters	**–** Deterministic hyper-parameters
APTA	P_D	$P_{D\mid\boldsymbol{\delta},\boldsymbol{\sigma}}(\boldsymbol{\delta},\boldsymbol{\sigma})$	$\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\delta},\boldsymbol{\sigma})$	$(\boldsymbol{\Delta},\boldsymbol{\Sigma}) \sim h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}(\boldsymbol{\delta},\boldsymbol{\sigma})$	$\left[-\boldsymbol{\delta}^{(\max)},\boldsymbol{\delta}^{(\max)}\right]$ $\left[\boldsymbol{\sigma}^{(\min)},\boldsymbol{\sigma}^{(\max)}\right]$ $\left[\boldsymbol{C}_p^{(\min)},\boldsymbol{C}_p^{(\max)}\right]$
Reliability	$\widetilde{P}_{\mathrm{f}}$	$P_{\mathrm{f}}(\boldsymbol{\theta})$	$\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta})$	$\boldsymbol{\Theta} \sim h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\xi})$	$\boldsymbol{\xi}$

The last equation, Eq. (12c), implies the need to develop a numerical strategy to estimate $\widetilde{P}_{\mathrm{f}}$. It appears that it can be numerically solved by two different approaches: a *nested approach* and an *augmented approach* [24].

To put it simple, the nested approach is similar to a *double-loop* Monte Carlo sampling strategy which requires to evaluate two different quantities: the first (inside) loop estimates the conditional failure probability given in Eq. (11) by integrating over $\mathcal{D}_{\mathbf{X}}$. The second (outside) loop estimates the predictive failure probability given in Eq. (12b) by integrating over $\mathcal{D}_{\boldsymbol{\Theta}}$. In terms of sampling strategy, the nested



(a) Nested approach (3 × 200 samples/cloud) (b) Augmented approach (600 samples)

Figure 3 – Nested vs. augmented sampling strategies (600 samples in total) for $\mathrm{X}_1 \sim \mathcal{N}(\mu_{\mathrm{X}_1} = 7, \sigma_{\mathrm{X}_1} = 5/\sqrt{3})$, $\mathrm{X}_2 \sim \mathcal{N}(\Theta, \sigma_{\mathrm{X}_2} = 2/\sqrt{3})$ and one uncertain parameter $\Theta = \mu_{\mathrm{X}_2} \sim \mathcal{N}(2, 1.5)$

approach can be illustrated by the trivial example shown in Figure 3a. This approach has been used in literature in several contexts [25, 26].

The augmented approach relies on the definition of an augmented space through the vector $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{\Theta}, \mathbf{X})^\top$ [27]. The idea is to sample jointly over both domains while respecting the conditioning of $\mathbf{X}$ on $\mathbf{\Theta}$ though. The sampling strategy can be illustrated as in Figure 3b. It thus reveals that, at least, both strategies are equivalent, but the augmented one offers better space-filling properties than the nested one and allows to reduce the computational cost.

In this paper, we compare the use of an augmented approach to estimate the defect probability by the APTA method with the traditional formulation in [4] based on a nested sampling strategy coupled with FORM. Again, under the consideration of a set of nonlinear limit-state functions, FORM approximation may lead to strong errors in the defect probability estimation. Thus, using a simulation method could be the only way to assess the conditional defect probability. This double-loop approach could be unnecessarily expensive and could be replaced by an augmented one.

To summarize the different approaches, three generic algorithms are provided below. Algorithm 1 illustrates a double-loop nested approach, based on two Monte Carlo sampling loops. This purely nested approach is the one which is the most encountered in engineering practice due to its easy-to-implement aspect and robustness with respect to nonlinearities of the tolerance chain function or to the input dimension. However, depending on the rareness of the defect probability, the simulation budget to achieve convergence can be an issue. Moreover, if the tolerance chain function is a time consuming computer code, this nested approach becomes untractable. Algorithm 2 illustrates the orignal APTA, as formulated in [4]. The inner loop is replaced by a FORM analysis. The main advantage is that one loop is removed which makes the algorithm less computationally demanding. Nevertheless, the main drawbacks first concern the conditional defect probability estimation. In some specific cases, FORM may be not licit. Then, the outer sampling loop may be time consuming regarding the accuracy one wants over the V_D space. As a remark, one can notice the preliminary isoprobabilistic transformation phase, $\mathbf{u} = T^{iso}(\mathbf{x})$, inherent to any FORM analysis. Such a phase is not discussed here in details, but we can just mention that in this case, the *Nataf transformation* is most often applied [28]. Finally, Algorithm 3 proposes an augmented version of the previous APTA algorithm. In this algorithm, the loops have disappeared. The augmented formulation of the problem through the vector $\mathbf{Z} = (\mathbf{\Theta}, \mathbf{X})^\top$

enables to solve the problem using any approximation method (here, FORM and Multi-FORM) or advanced simulation method (here, SS, which requires to defined a number of simulations N_{sim} per step). That means that either nonlinear or costly tolerance chain functions can be efficiently evaluated in this framework. In this case, all the problem is set by defining an adapted isoprobabilistic transformation $\mathbf{u} = T^{iso}(\mathbf{z})$ on the augmented vector. Here, the *Rosenblatt transformation* [29] has to be first applied to the stochastic distribution parameters (mean shift and standard deviation for the tolerance analysis) and then to the basic input random variables, conditonally to the realizations of the parameters. The final defect probability is then computed and obtained as the output of the chosen algorithm (which can be any approximation or simulation method) replacing the symbols *** in Algorithm 3. In the case of a simulation method which does not require any isoprobabilistic transformation (e.g., CMC or some versions of IS), Algorithm 3 just reduces in a single vectorized simulation step. As a final remark, one should notice that, in this augmented formulation, we do not have access to the conditional defect probability anymore.

Algorithm 1: Nested CMC for tolerance analysis

Start;
Define: $h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}$, $f_{\mathbf{X}}$, $N_{\mathbf{x}}$, $N_{\boldsymbol{\delta},\boldsymbol{\sigma}}$;
For $k = 1 : N_{\boldsymbol{\delta},\boldsymbol{\sigma}}$;
  Sample $\left(\boldsymbol{\Delta}^{(k)},\boldsymbol{\Sigma}^{(k)}\right)$;
  For $i = 1 : N_{\mathbf{x}}$;
    Sample $\mathbf{X}^{(i)} \mid \left(\boldsymbol{\delta}^{(k)},\boldsymbol{\sigma}^{(k)}\right)$;
    Evaluate: $g(\mathbf{x}^{(i)})$;
  Get $\widehat{P}^{(k)}_{\mathrm{D}|\boldsymbol{\delta}^{(\mathrm{k})},\boldsymbol{\sigma}^{(\mathrm{k})}} = \frac{1}{N_{\mathbf{x}}}\sum_{i=1}^{N_{\mathbf{x}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X}^{(i)})$;
$\widehat{\widehat{P}}_{\mathrm{D}} = \frac{1}{N_{\boldsymbol{\delta},\boldsymbol{\sigma}}}\sum_{k=1}^{N_{\boldsymbol{\delta},\boldsymbol{\sigma}}} \widehat{P}^{(k)}_{\mathrm{D}|\boldsymbol{\delta}^{(\mathrm{k})},\boldsymbol{\sigma}^{(\mathrm{k})}}$

Algorithm 2: Nested APTA with FORM [4]

Start;
Define: $h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}$, $f_{\mathbf{X}}$, $N_{\boldsymbol{\delta},\boldsymbol{\sigma}}$;
For $k = 1 : N_{\boldsymbol{\delta},\boldsymbol{\sigma}}$;
  Sample $\left(\boldsymbol{\Delta}^{(k)},\boldsymbol{\Sigma}^{(k)}\right)$;
  Define: $\mathbf{u} = T^{iso}(\mathbf{x})$;
  Solve a FORM analysis;
  Get $\widehat{P}^{(k)}_{\mathrm{D}|\boldsymbol{\delta}^{(\mathrm{k})},\boldsymbol{\sigma}^{(\mathrm{k})}} = p_{\mathrm{f,FORM}}$;
$\widehat{\widehat{P}}_{\mathrm{D}} = \frac{1}{N_{\boldsymbol{\delta},\boldsymbol{\sigma}}}\sum_{k=1}^{N_{\boldsymbol{\delta},\boldsymbol{\sigma}}} \widehat{P}^{(k)}_{\mathrm{D}|\boldsymbol{\delta}^{(\mathrm{k})},\boldsymbol{\sigma}^{(\mathrm{k})}}$

Algorithm 3: Augmented APTA

Start;
Define: $h_{\boldsymbol{\Delta},\boldsymbol{\Sigma}}$, $f_{\mathbf{X}}$,(N_{sim}/step);
Define: $\mathbf{u} = T^{iso}(\mathbf{z})$;
Augmented * analysis**;
$\widehat{\widehat{P}}_{\mathrm{D}} =$ **Output of *****

N.B.: *** stands for any method such as FORM, SORM, SS, etc.

The following section aims at illustrating the benefits of the augmented framework for solving statistical tolerancing problems with the APTA method on a two-part assembly test-case.

4 Numerical applications

The following numerical application has been implemented in Matlab® and performed using the open source toolbox FERUM v4.1 [30].

4.1 Two-part assembly illustration

As an illustrative example, we are considering a two-part-assembly as presented in Figure 4. This example is the same as the one presented as reference example in [4]. We use it here as a benchmark to compare results obtained by the traditional APTA and those obtained by the use of the augmented formulation of this method.

The functional requirement can be expressed with the following linear tolerance chain function:

$$Y = f(\mathbf{X}) = X_1 + X_2 \tag{13}$$

$$\text{s.t.} \quad Y \in [9.5, 10.5]\,. \tag{14}$$

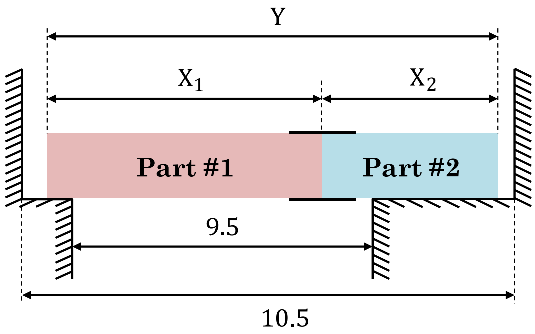


Figure 4 – Linear two-part assembly.

The customer specifies to the manufacturer a production quality requirement of 5 ppm, which means that $P_D = \mathbb{P}\left[Y \notin [9.5, 10.5]\right] \leq 5 \times 10^{-6}$. The process specifications can be found in Table 2. A general input probabilistic model is given in Table 3. In the following, the notations $\boldsymbol{\delta} = (\delta_1, \delta_2)^\top$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^\top$ are used. If we refer to the original paper [4], we could consider three different scenarios:

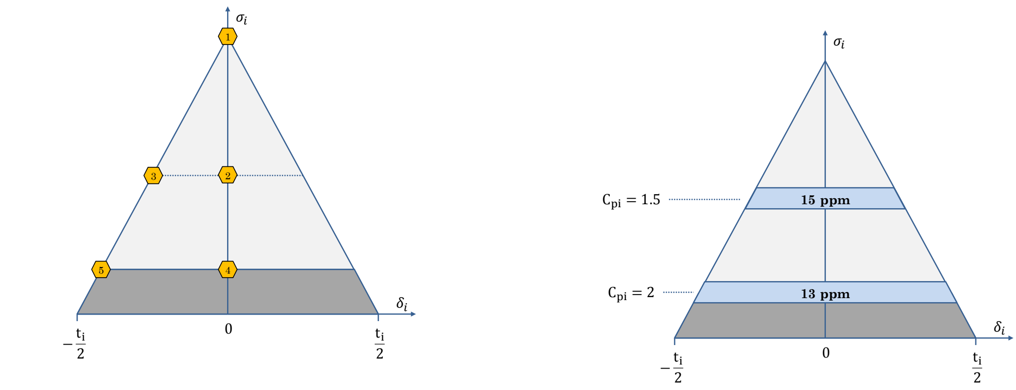
- Case #0 - Static assumptions: the distribution parameters are assumed to be fixed at some deterministic values $\boldsymbol{\Delta} = \boldsymbol{\delta}^0$ and $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^0$. The results for this scenario are just recalled in Figure 5a for the sake of clarity but are no more discussed in the following.
- Case #1 - Uniform mean shift and fixed standard deviation: in this case, $\boldsymbol{\Delta} \sim \mathcal{U}\left(\left[-\boldsymbol{\delta}^{(\max)}; \boldsymbol{\delta}^{(\max)}\right]\right)$ and $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^0$. The results for this scenario are just recalled in Figure 5b for the sake of clarity but are no more discussed in the following.
- Case #2 - Uniform mean shift and uniform standard deviation: in this case, $\boldsymbol{\Delta} \sim \mathcal{U}\left(\left[-\boldsymbol{\delta}^{(\max)}; \boldsymbol{\delta}^{(\max)}\right]\right)$ and $\boldsymbol{\Sigma} \sim \mathcal{U}\left(\left[\boldsymbol{\sigma}^{(\min)}; \boldsymbol{\sigma}^{(\max)}\right]\right)$.

Following Eqs. (6) and (7), one can consider either the standard deviation or the capability as random. In the following, according to a common manufacturing engineering practice, we will sample over the capabilities such that $\boldsymbol{C}_p \sim \mathcal{U}\left(\left[\boldsymbol{C}_p^{(\min)}; \boldsymbol{C}_p^{(\max)}\right]\right)$ for Case #2.

In the following, we will only focus and analyze simulations and results concerning Case #2 which is the most complete test-case since both mean shift and standard deviation are varying.

Table 2 – Process specifications for the two-part example.

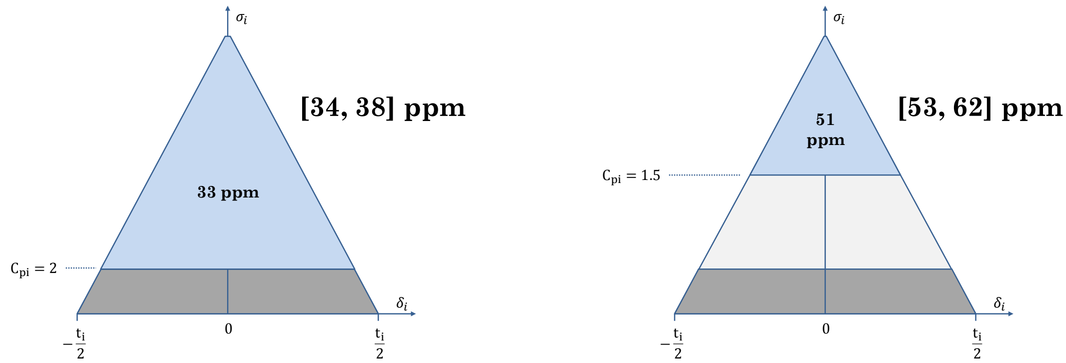
X_i	T_i	t_i	$C_{pi}^{(r)}$	$C_{pki}^{(r)}$
X_1	6	$1/(2\sqrt{2})$	1	1
X_2	4	$1/(2\sqrt{2})$	1	1



(a) Results for Case #0: (1) ≡ 318 ppm, (2) ≡ 0 ppm, (3) ≡ 521 ppm, (4) ≡ 0 ppm, (5) ≡ 1551 ppm.

(b) Results for Case #1.

Figure 5 – Bibliography results for the two-part example for Case #0 and Case #1.



(a) Results for $C_{pi}^{(\max)} = 2$.

(b) Results for $C_{pi}^{(\max)} = 1.5$.

Figure 6 – Bibliography results for the two-part example for Case #2 and comparison with the results obtained by the augmented approach (in brackets).

Table 3 – Input probabilistic model for the two-part example.

Variable	Distribution	Mean shift	Std
X_1	Normal	δ_1 ***uncertain***	σ_1 ***uncertain***
X_2	Normal	δ_2 ***uncertain***	σ_2 ***uncertain***
Δ_1	Uniform	$-\delta_1^{(\max)}$	$\delta_1^{(\max)}$
Σ_1	Uniform	$\sigma_1^{(\min)}$	$\sigma_1^{(\max)}$
Δ_2	Uniform	$-\delta_2^{(\max)}$	$\delta_2^{(\max)}$
Σ_2	Uniform	$\sigma_2^{(\min)}$	$\sigma_2^{(\max)}$

4.2 Simulation results

Table 4 – Simulation results for the two part example.

Ref. [4]	$C_{pi} = 2$ **33 ppm**			$C_{pi} = 1.5$ **51 ppm**		
	Estimate	Var	# g-calls	Estimate	Var	# g-calls
Nested APTA	34 ppm	4×10^{-15}	10^6 FORM	54 ppm	4×10^{-15}	10^6 FORM
Augmented FORM	38 ppm	–	$436 + 436 = 872$	62 ppm	–	$476 + 176 = 652$
Augmented Multi-FORM [16]	38 ppm	–	$462 + 540 = 1002$	62 ppm	–	$176 + 202 = 378$
Augmented SS	34 ppm	1×10^{-10}	10173	53 ppm	1.8×10^{-10}	10152

Table 4 illustrates both reference results from the original article [4] and simulations obtained by Nested APTA (i.e. Nested FORM) to find the original results again. Then, results obtained by the augmented approach coupled to three reliability methods to estimate the defect probability are proposed with their respective number of calls to the limit-state function. The mean estimate and the variance of the estimation are provided. These statistical results have been obtained by averaging a hundred replicates of the different algorithms.

One can see that traditional Nested APTA requires 10^6 FORM analyses (i.e. around 24×10^6 g-calls) to get the same results as those of reference [4]. On the one hand, all the methods give accurate results even if the Augmented FORM and the Augmented Multi-FORM overestimate the defect probability. On the other hand, all the augmented-based approaches seriously challenge the original Nested APTA since only a very small number of code evaluations is required. The sums which appear in the "# g calls" columns correspond to the sum of the number of calls for each failure point corresponding to the two hyperplanes.

This test-case reveals how efficient the formulation of some classical reliability approaches (FORM, Multi-FORM and Subset Simulations) is in terms of number of calls to the limit-state function (which is directly linked to the computational cost) and in terms of accuracy.

5 Conclusion

This paper deals with the problem of estimating a defect probability in statistical tolerancing problems. The major issue remains the fact that one has to face a bi-level uncertainty. Indeed, part manufacturing implies that part dimensions vary intrinsically, but statistical process parameters (mean shift value and standard deviation) also vary from one batch to another. Thus, the defect probability estimation has to take these two uncertainty levels into account. This paper proposes to couple an efficient Bayesian sampling strategy, the *augmented approach*, and an existing method in literature, APTA, for estimating defect probability in tolerance analysis problems. The application of the methodology on an academic test-case demonstrated numerical accuracy of the coupling strategy compared to the traditional nested estimation using the original version of APTA.

A future research track consists in extending this work to problems involving a nonlinear tolerance chain function. Another track could be to derive and estimate some sensitivities w.r.t. the statistical process parameters. These perspectives are left for future research.

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