COUPLING OF MFS AND ANM FOR SOLVING NONLINEAR ELASTICITY PROBLEMS

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Abstract :

In this work, we propose an algorithm that combines the Method of Fundamental Solutions (MFS) with the Asymptotic Numerical Method (ANM) for solving nonlinear elasticity problems. The ANM allows one to transform the nonlinear differential equations into a sequence of linear differential equations having the same tangent operator. Each linear resulting problem is then solved by using MFS. This last technique belongs to meshless collocation methods which has attracted considerable attention in recent years. It consists in constructing the solution by considering a linear combination of fundamental solutions of the differential operator. Regularization methods such as Truncated Singular Value Decomposition (TSVD) associated with the Generalized Cross Validation (GCV) criterion have been used to solve the ill-conditioned resultant linear systems. Two examples of nonlinear elasticity problems have been studied and have shown the robustness of the proposed algorithm.

Keywords : Method of Fundamental Solutions, Asymptotic Numerical Method, nonlinear elasticity, regularization methods.

1 Introduction

In this work, we propose to combine the Asymptotic Numerical Method (ANM) and a meshless technique based on the Method of Fundamental Solutions (MFS) for solving nonlinear elasticity problems. The ANM is considered as an efficient tool to solve nonlinear partial differential equations without need of any iteration procedure. It consists in expanding the variables into Taylor series that are truncated at rather large orders. This allows one to transform the nonlinear problem into a sequence of linear ones and to obtain a large part of the solution branch. Furthermore, as the step length is limited by the convergence radius of the series, a continuation procedure is performed to obtain the whole solution branch in a step by step manner. Many applications have established the robustness of this method for nonlinear solid and fluid mechanics, nonlinear vibrations, contact, large displacements and large rotations, plasticity and other fields in physics [4, 5, 6, 7].

Generally, the resulting linear problems issue of the perturbation technique are solved by using finite element method. Recently ANM technique has been associated to meshless methods and particularly to the method of fundamental solutions [1, 2, 3]. This method belongs to collocation techniques and does not require meshes as for finite element method. Due to its simplicity of implementation, the MFS seems to be a more and more attractive tool to solve linear and nonlinear differential equations in many computational mechanics area. Introduced first by Kupradze and Aleksidze [8], the MFS has been proven also to be a very efficient method to study some linear elasticity problems. Indeed, the linear elasticity Cauchy problem is discussed in [11] and the non-homogeneous linear elasticity equations are treated in [10]. In reference [12], Marin et al. have employed MFS with regularization techniques to study the inverse boundary value problems in three dimensional steady state linear thermo-elasticity. Moreover, the MFS has been extended to solve some nonlinear problems in many engineering fields. It was mainly combined with classical iterative methods as Newton–Raphson one or variants [20, 21, 22, 23, 24]. In references [1, 2], authors have coupled MFS with ANM for solving nonlinear problems and for analysis of bifurcation in [3]. However, only a few researchers have investigated Method of Fundamental Solutions MFS to solve nonlinear elasticity problems. Among them, we note the works of Naffa et al. [25, 26] which have used an iterative method associated with Radial Basis Functions (RBF) to solve the nonlinear differential equations governing large deflection of thin plates.

To our knowledge, the MFS has not been yet applied to a nonlinear elasticity problems. For that, we propose in this work to extend it for solving nonlinear elasticity problems involving large deformations by conjunction with ANM. The regularization based on Truncated Singular Value Decomposition (TSVD) [13, 14] is employed to solve the ill-conditioned resultant linear system while the regularization parameter is chosen by the Generalized Cross Validation (GCV) criterion [15].

The layout of this paper is as follows. In section 2, the mathematical formulation for large deformation problems in 2D elastostatic framework is given. The description of ANM is presented in section 3. Spatial discretization using MFS is discussed in section 4 and in section 5, we illustrate briefly the regularization method and selection criteria. On the other hand side, numerical examples involving large deformation problems are provided to show the efficiency and accuracy of the proposed algorithm.

2 Governing equations

This section presents the basic equations of nonlinear elasticity in strong form. The nonlinear elasticity is an important nonlinear problem in computational mechanics. The strong form of the boundary value problem for two-dimensional nonlinear elasto-static is as follows:

$$\begin{cases} \{\gamma\} &= ([II] + \frac{1}{2}[A(\theta)])\{\theta\} & in\ \Omega \\ \{S\} &= [D]\{\gamma\} & in\ \Omega \\ \{\Pi\} &= ([III] + [B(\theta)])\{S\} & in\ \Omega \\ [div]\,\{\Pi\} &= \{0\} & in\ \Omega \\ [N]\,\{\Pi\} &= \lambda\{T^d\} & in\ \partial\Omega_f \\ \{U\} &= \lambda\{U^d\} & in\ \partial\Omega_u \end{cases} \tag{1}$$

The open set $\Omega \subset R^2$ with smooth boundary $\partial\Omega$ represents a bounded reference configuration for the continuum body. The boundary $\partial\Omega$ is decomposed into two parts $\partial\Omega_u$ and $\partial\Omega_f$. The displacements $\{U^d\}$ is prescribed on $\partial\Omega_u$ and the traction $\{T^d\}$ is prescribed on $\partial\Omega_f$ that gives the intensity of the applied load. The components of outward unit normal vector are regrouped in the matrix $[N]$ and λ is a scalar parameter. In the case of large deformations, $\{\gamma\}$ represents the Green–Lagrange strain tensor which is expressed versus the deformation gradient $\{\theta\}$. In the nonlinear elasticity theory, the Lagrangian description is the most formulation used for this kind of problems which permits to determine the exact transformation between the reference and current configurations. $\{\Pi\}$ and $\{S\}$ represent respectively the 1^{st} and 2^{nd} Piola-Kirchhoff stress tensors which are used to express the pseudo-stress relative to the reference configuration. The constitutive matrix $[D]$ for a homogeneous and isotropic elastic material is written as follows:

$$[D] = \frac{\overline{E}}{1-\overline{\nu}^2}\begin{bmatrix} 1 & \overline{\nu} & 0 \\ \overline{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\overline{\nu}}{2} \end{bmatrix} \tag{2}$$

were $\overline{E} = E, \overline{\nu} = \nu$ for the plane stress condition and $\overline{E} = E/(1-\nu^2), \overline{\nu} = \nu/(1-\nu^2)$ for the plane strain condition, E and ν are respectively the Young's modulus and the Poisson's ratio. The matrices $[A(\theta)]$, $[B(\theta)]$, $[III]$ and $[II]$ are given by:

$$\begin{gathered} [A(\theta)] = \begin{bmatrix} U_{1,1} & 0 & U_{2,1} & 0 \\ 0 & U_{1,2} & 0 & U_{2,2} \\ U_{1,2} & U_{1,1} & U_{2,2} & U_{2,1} \end{bmatrix}; [B(\theta)] = \begin{bmatrix} U_{1,1} & 0 & U_{1,2} \\ 0 & U_{2,2} & U_{2,1} \\ 0 & U_{1,2} & U_{1,1} \\ U_{2,1} & 0 & U_{2,2} \end{bmatrix}; \\ [II] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; [III] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{gathered} \tag{3}$$

The quantities U_1 and U_2 represent the components of the displacement vector $\{U\}$ and $U_{i,j}$ indicates the derivative of the component U_i with respect to j^{th} variable. The system of equation (1) is nonlinear and therefore requires linearization algorithms. Among these algorithms, we find the iterative incremental methods which are very expensive in computation time. In this work, algorithms based on the ANM are other alternatives for solving nonlinear equations such as system (1). These algorithms are explained in detail in the next section.

3 High order algorithm

The high order algorithm, often known as ANM (Asymptotic Numerical Method) in the literature, is a numerical solver for nonlinear equations. The conventional tangent approximations are replaced by power series truncated to a high order. The ANM transforms the problem (1) into a sequence of linear problems which are solved using a meshless method such that MFS. The use of the asymptotic expansion requires to write the system (1) in a quadratic form. Some details of this procedure are given in the reference [16]. We use the mixed vector $\{\mathbb{U}\} = \{\{\Pi\}, \{S\}, \{\gamma\}, \{\theta\}, \{U\}\}$ which represents the fundamental physical fields of the problem. The variables $\{\mathbb{U}\}$ and λ are developed using an asymptotic expansion truncated at order P with respect to a path parameter "a" in the neighborhood of a known starting solution $(\{\mathbb{U}_0\}, \lambda_0)$. Thus, we can write:

$$\begin{Bmatrix} \{\mathbb{U}\} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \{\mathbb{U}_0\} \\ \lambda_0 \end{Bmatrix} + \sum_{i=1}^{P} a^i \begin{Bmatrix} \{\mathbb{U}_i\} \\ \lambda_i \end{Bmatrix} \tag{4}$$

By introducing eqn (4) into eqn (1) and equating like powers of "a", we obtain the following set of linear mixed problems:

- **Order** 1

$$\left\{ \begin{array}{lcl} \{\gamma_1\} & = & [F(\theta_0)]\{\theta_1\} \\ \{S_1\} & = & [D]\{\gamma_1\} \\ \{\Pi_1\} & = & [H(\theta_0)]\{S_1\} + [\widehat{S}_0]\{\theta_1\} \\ [div]\,\{([F(\theta_0)][D][H(\theta_0)] + [\widehat{S}_0])\{\theta_1\}\} & = & \{0\} \\ [N]\,\{([F(\theta_0)][D][H(\theta_0)] + [\widehat{S}_0])\{\theta_1\}\} & = & \lambda_1\{T^d\} \\ \{U_1\} & = & \lambda_1\{U^d\} \end{array} \right. \tag{5}$$

- **Order** $2 \leq k \leq P$

$$\left\{ \begin{array}{lcl} \{\gamma_k\} & = & [F(\theta_0)]\{\theta_k\} + \{\gamma_k^{nl}\} \\ \{S_k\} & = & [D]\{\gamma_k\} \\ \{\Pi_k\} & = & [H(\theta_0)]\{S_k\} + [\widehat{S}_0]\{\theta_k\} + \{\Pi_k^{nl}\} \\ [div]\,\{([F][D][H] + [\widehat{S}_0])\{\theta_k\}\} & = & -[div]\{[F][D]\{\gamma_k^{nl}\} + \{\Pi_k^{nl}\}\} \\ [N]\,\{([F][D][H] + [\widehat{S}_0])\{\theta_k\}\} & = & \lambda_k\{T^d\} - [N]\{[F][D]\{\gamma_k^{nl}\} + \{\Pi_k^{nl}\}\} \\ \{U_k\} & = & \lambda_k\{U^d\} \end{array} \right. \tag{6}$$

where $[F(\theta_0)] = [II] + [A(\theta_0)]$, $[H(\theta_0)] = [III] + [B(\theta_0)]$, $\{\gamma_k^{nl}\} = \dfrac{1}{2}\sum_{r=1}^{k-1}[A(\theta_r)]\{\theta_{k-r}\}$ and $\{\Pi_k^{nl}\} = \sum_{r=1}^{k-1}[B(\theta_r)]\{S_{k-r}\}$. The term $[B(\theta_i)]\,\{S_0\}$ is transformed to $[\widehat{S}_0]\,\{\theta_i\}$ to reveal the unknown $\{\theta_i\}$; with the matrix $[\widehat{S}_0]$ contains the stress of the starting solution defined as:

$$[\widehat{S}_0] = \begin{bmatrix} S_{11}^0 & S_{12}^0 & 0 & 0 \\ 0 & 0 & S_{12}^0 & S_{22}^0 \\ S_{12}^0 & S_{22}^0 & 0 & 0 \\ 0 & 0 & S_{11}^0 & S_{12}^0 \end{bmatrix} \tag{7}$$

where the terms S_{ij}^0 are the components of the second stress tensor. In this formulation, only the displacement vector is discretized. For this, a substitution method is used to condense the different equations (6.a), (6.b) and (6.c) into equilibrium ones (6.d) and (6.f). Note that the set of equations (6.d) , (6.e) and (6.f) is a system of linear equations having the same tangent matrix for all the orders depending on the starting geometry and stress $\{\theta_0\}$ and $\{S_0\}$. This procedure involves calculating the term of order P of the series (4) versus previous orders. An additional equation called the auxiliary equation must be added to define the path parameter "a". This path parameter "a" can be defined in three different ways: either as the load increment $(\lambda - \lambda_0)$, or as a component of the displacement increment $(\{U\} - \{U_0\})$ or an arc length [17]. However, this last case is very used when the branch presents limit points. According this idea, we can identify the path parameter "a" as follow:

$$a = \langle(\{U\} - \{U_0\}), \{U_1\}\rangle + \langle(\lambda - \lambda_0), \lambda_1\rangle \tag{8}$$

where $< \cdot, \cdot >$ is the euclidian scalar product. The equation (8) provides an adaptative path parameter which enables one to describe the branch. By introducing the series (4) into (8) and equating like powers of "a", we obtain the set of single equations. Within the framework of the ANM, the series permit us to obtain a large part of the branch solution by inverting or decomposing only once the stiffness tangent matrix. This procedure can be considered as a high-order predictor which, generally, does not require the correction phase. The continuation technique permits us to determine the entire branch solution [17]. Each solution branch is characterized by a starting point and a validity range $[0, a_{max}]$. In this paper, the validity range of the series representation can be estimated by a criterion defined in [16, 17] which is expressed as a function of the truncation order P, a given tolerance ε and a right-hand side $\{F^{nl}_{P+1}\}$ which contains the nonlinear terms calculated at the previous orders defined in the following section. The maximum value of the parameter "a" is defined as follow:

$$a_{max} = \left(\frac{\varepsilon}{\parallel \{F^{nl}_{P+1}\} \parallel}\right)^{\dfrac{1}{P+1}} \tag{9}$$

The solution $\{\{\mathbb{U}(a_{max})\}, \lambda(a_{max})\}$ is a new starting solution for the following step. The complete solution branch is obtained step by step via the continuation technique.

4 Meshless numerical modeling

The Method of Fundamental Solutions (MFS) is currently a useful method in numerical meshless-type discretization techniques for solving linear partial differential equations. In the MFS, the principal unknown $\{U\}$ of problems (5) and (6) is approximated by a sum of the homogeneous and the particular solutions. The homogeneous solution is defined as a linear combination of the fundamental solutions in terms of source points located outside the domain. The particular solution is written in the form of a linear combination of particular solutions in terms of collocation points. By integrating the MFS-MPS method with the Analogous Equations Method (AEM) to reveal the linear operator of the two-dimensional elasticity whose fundamental solution is previously known. Indeed, the system of linear equations defined in (5) and (6) contains an operator whose fundamental solution is unknown because of the presence of terms $[A(\theta_0)]$, $[B(\theta_0)]$ and $[\widehat{S}_0]$. Then, these terms are eliminated in the operator and considered as a right-hand side. In this procedure, the homogeneous (fundamental) solution is known and corresponds to that of the linear elasticity operator. So, the particular solution is calculated using the Dual Reciprocity Method (DRM) which uses the Radial Basis Functions (RBF) as an approximation of the right-hand side. This allows us to write the approximation as follow:

$$\{U(M_i)\} = \textstyle\sum_{j=1}^{N_s} [\widehat{U}^h(M_i, Q_j)] \begin{Bmatrix} \alpha^h_j \\ \beta^h_j \end{Bmatrix} + \textstyle\sum_{j=1}^{N} [\widehat{U}^p(M_i, M_j)] \begin{Bmatrix} \alpha^p_j \\ \beta^p_j \end{Bmatrix} \tag{10}$$

where $Q_j(X^j_1, X^j_2)$ and $M_i(x^i_1, x^i_2)$ are respectively the coordinates of N_s source points on the fictitious border and the coordinates of the N collocation points. $[\widehat{U}^h(M_i, Q_j)]$ represents the fundamental solution matrix of the two-dimensional linear elasticity operator. This matrix is given by the following formula [9, 11]:

$$\widehat{U}_{kl}^{h}(r_{ij}) \quad = \quad \frac{1}{8\pi\mu(1-\nu)}\left(-(3-4\nu)log(r_{ij})\delta_{kl} + \frac{(x_k^i - X_k^j)(x_l^i - X_l^j)}{r_{ij}^2}\right) \tag{11}$$

$[\widehat{U}^p(M_i, M_j)]$ is the matrix of particular solutions. The particular solutions are obtained using the Dual Reciprocity Method (DRM). The procedure for obtaining these particular solutions for a right-hand side is given in a similar way as in the BEM [19, 18]. The multi-quadric RBF type is considered and the particular solutions are obtained analytically from this function. Thereafter, we propose to rewrite the expression (10) in a compact form in which the matrices of the particular and homogeneous solutions are concatenated in a single matrix $[\widehat{U}]$ and the coefficients of linear combinations α_j^h, β_j^h, α_j^p and β_j^p are grouped into a single vector $\{X\}$ in the following form:

$$\{U(P_i)\} \quad = \quad [\widehat{U}(P_i)]\{X\} \tag{12}$$

By injecting the approximation (12) into the set of equations (5.d), (5.e), (5.f), (6.d), (6.e) and (6.f), we obtain a linear algebraic system which is written as the following form:

$$\begin{array}{llcl} Order & 1 & : & [K_T]\{X_1\} = \lambda_1\{F\} \\ Order & 2 \le k \le P & : & [K_T]\{X_k\} = \lambda_k\{F\} + \{F_k^{nl}\} \end{array} \tag{13}$$

where the matrix $[K_T]$ and the vectors $\{F\}$, $\{F_k^{nl}\}$ are defined by:

$$[K_T] = \begin{bmatrix} [div]\left(([F][D][H] + [\widehat{S}_0])[grad][\widehat{U}(P_i)]\right) \\ [N]\left(([F][D][H] + [\widehat{S}_0])[grad][\widehat{U}(P_i)]\right) \\ \left[\widehat{U}(P_i)\right] \end{bmatrix} \begin{array}{l} P_i \in \Omega \\ P_i \in \partial\Omega_f \\ P_i \in \partial\Omega_u \end{array} \tag{14}$$

$$\{F_k^{nl}\} = \left\{ \begin{array}{c} -[div]\left\{[F][D]\{\gamma_k^{nl}(P_i)\} + \{\Pi_k^{nl}(P_i)\}\right\} \\ -[N]\left\{[F][D]\{\gamma_k^{nl}(P_i)\} + \{\Pi_k^{nl}(P_i)\}\right\} \\ \{0\} \end{array} \right\} \begin{array}{l} P_i \in \Omega \\ P_i \in \partial\Omega_f \\ P_i \in \partial\Omega_u \end{array} \tag{15}$$

$$\{F\} = \left\{ \begin{array}{c} \{0\} \\ \{T^d\} \\ \{U^d\} \end{array} \right\} \begin{array}{l} P_i \in \Omega \\ P_i \in \partial\Omega_f \\ P_i \in \partial\Omega_u \end{array} \tag{16}$$

5 Regularization methods

The MFS system of linear algebraic equations (13) is ill-conditioned. This implies that the standard numerical methods for solving this system are not effective. Generally, the resolution of the ill-conditioned linear system requires a particular processing in order to minimize its sensitivity to disturbances whence the regularization methods find their utility for the resolution of this kind of problem. The Singular Value Decomposition (SVD) is a fundamental tool for studying the conditioning of the matrix $[K_T]$. This technique is applied in the regularization method TSVD and in the Tikhonov method. In this paper, we use the TSVD regularization [13, 14] which consists in truncating the decomposition into singular values of the matrix $[K_T]$. The small singular values of $[K_T]$ are eliminated and searched the solution that minimizes the least square of the system (13) which is written as follow:

$$\{X_i^\eta\} = \sum_{r=1}^{\eta} \frac{<\Lambda_r>\{b_i\}}{s_r}\{w_r\} \quad , \quad 1<\eta<m \tag{17}$$

where m is the rank of the matrix $[K_T]$, η is the truncation parameter and the vector $\{b_i\} = \lambda_i\{F\} + \{F_i^{nl}\}$ represents the right-hand side of the system (15). The SVD decomposition of the matrix $[K_T]$ is written as:

$$[K_T] = [\Lambda][\Sigma][W] = \sum_{i=1}^{m} s_i\{\Lambda_i\}<w_i> \tag{18}$$

The matrices $[\Lambda]$ and $[W]$ are orthogonal; with $\{\Lambda_i\}$ and $\{w_i\}$ are the i^{th} columns respectively of $[\Lambda]$ and $[W]$, $[\Sigma]$ is a diagonal matrix containing the singular values s_i strictly positive such that $s_1 > s_2 > \cdots > s_m > 0$.

One of the difficulties of regularization methods remains in the choice of the optimal parameter values η. We proposed to couple the GCV criterion with the TSVD regularization for determining this parameter. The GCV method [15] is an empirical method and it is very simple to implement. It consists in searching for the regularization parameter η which minimizes the following function:

$$\upsilon(\eta) = \frac{\|[K_T]\{X_i^\eta\} - \{b_i\}\|}{tr([I] - [K_T][K_T]^T)^2} \tag{19}$$

6 Numerical results and discussions

6.1 Example 1: Bending of thin plate

In this example, we are interested to the resolution of nonlinear bending problem of a thin plate of length $L = 100mm$, width $l = 10mm$, made of a homogeneous, elastic and isotropic material of Young's modulus $E = 10000MPa$ and Poisson's ratio $\nu = 0.3$. The plate is embedded from its left end and subjected to a bending loading $\{T^d\} = {}^T\langle 0 \quad T_2^d \rangle$; with $(T_2^d = 1MPa)$ at the other end (see figure 1).

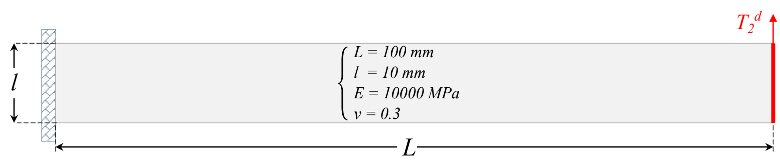

Figure 1: Thin plate under bending load

For numerical data, we take $N = 467$ collocation points arbitrarily distributed on the domain occupied by the plate (see Figure 2) and $N_s = 113$ source points on the fictitious boundary. The fictitious boundary is considered as a circle of radius $R = 70mm$ and of center $(x = 50, y = 0)$ which is the mass center of the plate. After several numerical tests, the optimal parameters of high order algorithm are chosen as: the truncation order $P = 15$ and the tolerance $\varepsilon = 10^{-6}$. The numerical analysis is based on the evolution of displacements of nodes $P1$ and $P2$, represented in figure 2, versus load parameter λ.

In figure 3, we represent the displacements of nodes $P1$ and $P2$ with respect to load parameter λ. These response curves of the nonlinear bending problem is obtained by the both algorithms using the MFS and FEM methods. These results show a good agreement between the both algorithms. The proposed

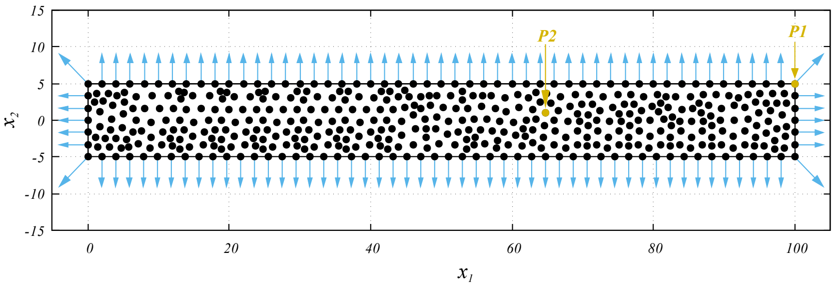

Figure 2: Distribution nodes and the normal vector at each point of the boundary $\partial\Omega$

algorithm requires five ANM-steps as shown in figure (3) i.e. five inversions of the tangent matrix to get the bending up to the value $U_2 = 78.8mm$ for the value of $\lambda = 91,04$.

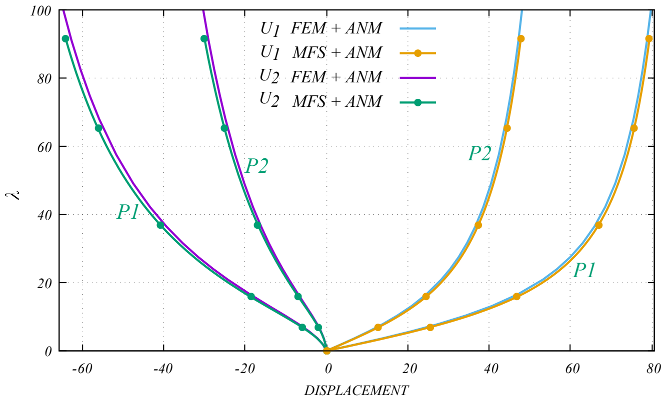

Figure 3: Load-displacement curve to the nodes $P1$ and $P2$

In this example, we use the regularization TSVD associated with the GCV criterion in order to reduce the hypersensitivity of the conditioning number.

6.2 Example 2: Buckling of thin plate

This example concerns the buckling of a thin plate simply supported at these both ends and subjected to a compressive load $\{T^d\} = {}^T\langle T_1^d \quad 0\rangle$; with ($T_1^d = 1MPa$) applied at its left end and a perturbation load $\{T^d\} = {}^T\langle 0 \quad T_2^d\rangle$ applied at the other end as shown in figure 4. The perturbation load is introduced in order to follow the bifurcated curve. The considered structure is of length $L = 100mm$, width $l = 10mm$, made of a homogeneous, elastic and isotropic material of Young's modulus $E = 10000MPa$ and Poisson's ratio $\nu = 0.3$. We adopt the same numerical data as in the first example. In this case, the results are recorded at the node $P2$.

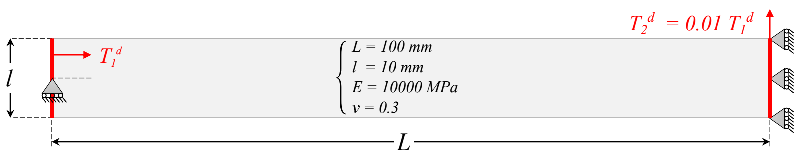

Figure 4: Buckling of a thin plate

In figure 5, we present the evolution displacements of node $P2$ versus the load parameter λ. From these results, we conclude the same remarks as in the first example. The proposed algorithm requires

17 ANM-steps because this problem is complicated than those of the first example.

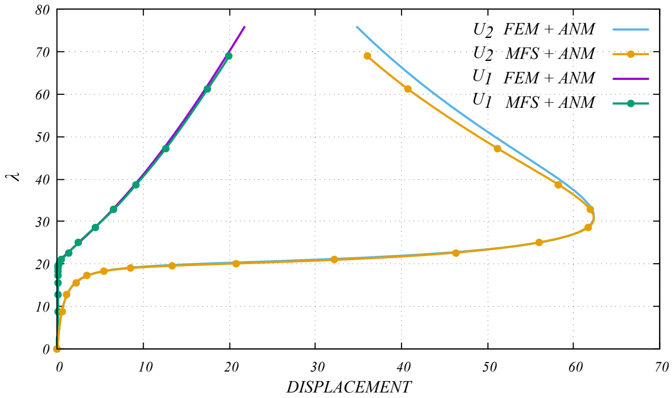

Figure 5: Load-displacement curve of the node $P2$

7 Conclusion

We have proposed, in this study, a numerical model which consists in coupling the ANM algorithm with meshless method to simulate the nonlinear elasticity problems. The meshless method used her is the Method of Fundamental Solutions (MFS) which is the true meshless method and have the concept of collocation approach. The obtained result illustrate the effectiveness of the proposed algorithm. In addition, we have used the TSVD regularisation associated with GCV criterion to overcome the difficulty associated with ill-conditioned linear system.

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