On the integration of additive manufacturing constraints in the framework of a NURBS-based topology optimisation method

G. COSTA[a], M. MONTEMURRO[a], J. PAILHES[a]

a.Institut de Mécanique et d'Ingénierie de Bordeaux (I2M) – Arts et Métiers ParisTech – Esplanade d'Arts et Métiers, 33400 Talence, France – email : Giulio.COSTA@ensam.eu; Marco.MONTEMURRO@ensam.eu; Jerome.PAILHES@ensam.eu.

Résumé :

Ce travail se focalise sur l'optimisation topologique des structures 2D : la méthode Solid IsotropicMaterialwithPenalisation (SIMP) est révisée et reformulée dans le cadre mathématique des fonctions NURBS (Non-Uniform Rational BSpline). Ce choix comporte plusieurs avantages : a)une surface NURBS est caractérisée par une zone de filtre définie de façon implicite ; b) le nombre de variables d'optimisation (à savoir les paramètres qui définissent la surface NURBS) est réduitvis-à-visde l'approche SIMP classique ;c) les contraintes non-conventionnelles liées au procédé de Fabrication Additive peuvent être facilement intégrées dans le processus d'optimisation topologique grâce au formalisme NURBS.L'efficacité de la méthode d'optimisation topologique proposée sera prouvéevia un benchmarkclassique.

Abstract :

This work focuses on the topology optimization (TO) of 2D structures: the Solid Isotropic Material with Penalisation (SIMP) method is revisited and reformulated within the mathematical framework of Non-Uniform Rational BSpline (NURBS) functions. Several advantages arise from such a choice: firstly, a NURBS surface allows for exploiting an implicitly defined filter zone; secondly, the number of optimisation variables (i.e. the parameters defining the NURBS surface) is relatively small when compared to the classical SIMP approach. Finally, the TO can be carried out by including non-conventional manufacturing constraints, as those related to the Additive Manufacturing (AM) technology.The proposed TO method is applied to a standard benchmark problem in this paper.

Mots clefs :NURBS, Topology Optimisation, Additive Manufacturing, SIMP

1 Introduction

Topology Optimisation (TO) is a well-known design tool that provides extremely efficient mechanical structures. Often, the mathematical optimum solution could involve a complicated geometry and topology: in some cases the optimised components cannot be fabricated through standard technologies. Nowadays, Additive Manufacturing (AM) seems to show all the requirements to achieve both optimised and manufacturable components in plastics or metal alloys (Guo and Leu, 2013). However, there are two keys factors preventing the link between TO and effective AM techniques. On the one hand, when an optimised solution is reassembled after TO analysisin a standard format file (".stp", ".igs" or ".stl"), a lot of time must be spent to obtain a connected and consistent geometry. So, further design operations in FEM or CAD software are hindered. On the other hand, despite its dimensional freedom, AM has intrinsic technological constraints which should be taken into account within TO analysis and not within a post-processing phase(Mirzendehdel and Suresh, 2016). Minimum and maximum member size have already been implemented respectively by (Poulsen, 2003) and (Guest, 2009) in the framework of the standard SIMP method: they are basic constraints for an AM process but they are not the only ones. As a matter of fact, the inhibition or the limitation of the support material is of paramount importance for AM structures(Kranz et al., 2015). Moreover,it is evident that further constraints capable of taking into account thermal effects and residual stresses, typical of AM, are required.

In this paper, an innovative TO methodology for 2D structures is proposed in order to overcome the aforementioned drawbacks and to get solutions that are designed for AM. The well-known SIMP method is modified by relating the fictitious density (or pseudo-density) field $\rho(\mathbf{x}) \in [0,1]$ to a suitable NURBS surface $\varphi(\mathbf{x})$ (Piegl and Tiller, 1997), where $\mathbf{x}$ is the position vector in the reference domain. Instead of assuming an unknown pseudo-density for each element of the underlying mesh, the number of variables is now defined by the value of the pseudo-density for each control point of the NURBS surface. Inspired by the idea of (Qian, 2013), when relating the SIMP density field to a suitable NURBS surface, many advantages occur: the first one is linked to the implicit filter zone that is defined by the blending functions local support. As consequence, artefacts typical of the SIMP method, such as the "checkerboard effect", as well as the mesh dependency are automatically overcome without establishing further filters. The present work goes beyond the analysis done by (Qian, 2013): the proposed strategy focuses on the design advantages, which can be got when the SIMP method is reformulated in the NURBS mathematical framework.
Firstly, it will be shown that, in the context of the classical TO benchmark problem dealing with the compliance minimisation subject to an imposed volume fraction (an equality optimisation constraint), the solutions exhibit clearly defined bounds. Volume constraints are met both in the TO process and in the post processing phase, where the resulting optimised geometry is handled by external software. Furthermore, the reconstruction phase for 2D structures is a completely automatic process. Another significant advantage is the independence of the design variables (i.e. the value of the pseudo-density at each control point of the NURBS surface) from the elements of the predefined mesh. Finally, the NURBS-based approach allows a mathematically well-defined description of the boundaries in terms of both local normal vector and local curvature radius, so it is possible to impose innovative constraints concerning the AM requirements. An unconventional constraint on the curvature radius I forecastfor the immediate future: it could enable the designer to manage both the smoothness of the boundaries and, indirectly, stress concentrations, which are typical in AM technologies.

The paper is structured as follows: in the second paragraph, the theoretical framework of the NURBS surfaces theory is briefly described. Then, in the third section the classic SIMP method is enhanced by means of the NURBS and the TO problem is stated as a constrained non-linear programming problem (CNLPP). The adopted numerical method is detailed in paragraph four. Section five illustrates a meaningful benchmark: in this background, the influence of the parameters defining the NURBS surface (number of control points, degrees of the surface) has been investigated. The sixth paragraph concludes this article with some critical discussion and remarkable future perspectives.

2 Theoretical framework of NURBS surfaces

In this section, the fundamentals of the NURBS surfaces theory are briefly recalled. It is noteworthy that, since only 2D problems are considered, a NURBS surface suffices to obtain a suitable representation of the density field as function of the spatial coordinates $x$and y defined over the design space.
According to the notation of (Piegl and Tiller, 1997), a NURBS surface is defined as follows:

$$\mathbf{S}(u,v) = \sum_{i=0}^{n_u}\sum_{j=0}^{n_v} R_{i,j}(u,v)\mathbf{P}_{i,j}, \tag{1}$$

where$R_{i,j}(u,v)$are the piecewise rational basis functions, which are related to the standard NURBS blending functions $N_{i,p}(u)$ and $N_{j,q}(v)$ by means of the relationship

$$R_{i,j}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)w_{i,j}}{\sum_{k=0}^{n_u}\sum_{l=0}^{n_v} N_{k,p}(u)N_{l,q}(v)w_{k,l}}. \tag{2}$$

In equations (1) and (2), $\mathbf{S}(u,v)$ is a bivariate vector-valued piecewise rational function, (u,v) are scalar dimensionless parameters both defined in the interval$[0,1]$, p and q are the NURBS degrees along u-direction and v-direction, respectively. $w_{i,j}$ are the weights and $\mathbf{P}_{i,j} = \{x_{i,j}, y_{i,j}, z_{i,j}\}$ the Cartesian coordinates of the control points, with $i \in [0, n_u]$ and $j \in [0, n_v]$. The net of $(n_u + 1) \times (n_v + 1)$ control points constitute the so-called control net. The blending functions are defined recursively by means of the Bernstein polynomials:

$$N_{i,0}(u) = \begin{cases} 1 \text{ if} U_i \leq u \leq U_{i+1}, \\ 0 \qquad \text{otherwise,} \end{cases} \tag{3}$$

$$N_{i,p}(u) = \frac{u - U_i}{U_{i+p} - U_i} N_{i,p-1}(u) + \frac{U_{i+p+1} - u}{U_{i+p+1} - U_{i+1}} N_{i+1,p-1}(u), \tag{4}$$

whereU_i is the $i-th$ component of the following non-periodic non-uniform knot vector

$$\mathbf{U} = \left\{\underbrace{0,\dots,0}_{p+1}, U_{p+1}, \dots, U_{m_u-p-1}, \underbrace{1,\dots,1}_{p+1}\right\}. \tag{5}$$

It is noteworthy that the size of the knot vector is $m_u + 1$,

$$m_u = n_u + p + 1. \tag{6}$$

Analogously, the $N_{j,q}(v)$ are defined on the knot vector $\mathbf{V}$, whose size is m_v:

$$\mathbf{V} = \left\{ \underbrace{0, \dots, 0}_{q+1}, V_{q+1}, \dots, V_{m_v-q-1}, \underbrace{1, \dots, 1}_{q+1} \right\}, \tag{7}$$

$$m_v = n_v + q + 1. \tag{8}$$

The knot vectors $\mathbf{U}$ and $\mathbf{V}$ are two non-decreasing sequences of real numbers that can be interpreted as two discrete collections of values of the dimensionless parameters $u$and v. As the control points, also the knot vectors components form a net. One basic property of the blending functions is the local support property: $N_{i,p}(u) = 0$ if u is outside the interval $[U_i, U_{i+p+1})$. Hence, it is evident that $R_{i,j}(u, v) = 0$ if (u, v) is outside the rectangle $[U_i, U_{i+p+1}) \times [V_j, V_{j+q+1})$, i.e. the local support associated to the control point $\mathbf{P}_{i,j}$. The local support property is of paramount importance to understand all the advantages of the NURBS formulation of the SIMP method in the context of TO. For a deeper insight in the NURBS theory, the reader is addressed to (Piegl and Tiller, 1997).

3 The NURBS-based Topology Optimisation method: mathematical formulation

The classic SIMP method is here revisited for the minimum compliance problem subject to an equality constraint on the volume fraction for a 2D problem. The reader is addressed to (Bendsøe and Sigmund, 2004) for a deeper insight into the matter.
In the framework of the proposed approach, the pseudo-density field characterising the SIMP method is related to a suitable NURBS scalar function. In the following, only Bspline functions have been employed for sake of simplicity, thus all the weights in equation (2) are equal to 1.
In the context of Bspline functions, the SIMP pseudo-density field writes:

$$\rho(u, v) = \sum_{i=0}^{n_u} \sum_{j=0}^{n_v} N_{i,p}(u) N_{j,q}(v) \bar{\rho}_{i,j}. \tag{9}$$

The shape of the Bspline is affected by the value of the pseudo-density at each control point, i.e. $\bar{\rho}_{i,j}$, as well as by the value of the other parameters involved into the definition of the Bspline scalar function, namely the degrees of the blending function, i.e. p and q, the number of control points (related to the parameters n_u and n_v) and the value of the knot vectors components, as illustrated in Eqs. (2) and (4). The dimensionless parameters u and v shown in Eq. (9) are related to the Cartesian coordinates of the global frame as:

$$u = \frac{x}{w}, \quad v = \frac{y}{h}, \tag{10}$$

where$w$ and h are the sizes of the 2D rectangular reference domain. In equation (9)$\bar{\rho}_{i,j}$ are the design variables of the NURBS-based SIMP method. They are collected in a column array $\boldsymbol{\xi}$ and suitable boundaries are imposed to satisfy the density field requirements for the TO problem:

$$\boldsymbol{\xi}^{\mathbf{t}} = \{\overline{\rho_{0,0}}, \dots, \overline{\rho_{n_u,0}}, \overline{\rho_{0,1}} \dots, , \overline{\rho_{n_u,1}}, \dots, \overline{\rho_{0,n_v}}, \overline{\rho_{n_u,n_v}}\},$$
$$\overline{\rho_{i,j}} \in [10^{-3}, 1]\ \forall i = 0, \dots, n_u, \forall j = 0, \dots, n_v. \tag{11}$$

Without loss of generality, in this work the two knots vector **U** and **V** are considered uniformly distributed in the interval $[0,1]$ and both the degrees of the blending functions and the number of control points are fixed a priori.

In this background, the TO problem can be stated for an attended volume fraction f as follow:

$$\min_{\boldsymbol{\xi}} l(\boldsymbol{\xi}),$$
$$\text{subject to:}$$
$$\begin{cases} E_{ijkl}\left(\rho(\boldsymbol{\xi})\right) = \rho(\boldsymbol{\xi})^{\alpha} E^{0}_{ijkl}, \\ \dfrac{V(\boldsymbol{\xi})}{V_{tot}} = \dfrac{\int_0^w \int_0^h \rho(\boldsymbol{\xi})\, dxdy}{wh} = f, \\ \mathbf{g}(\boldsymbol{\xi}) \leq \mathbf{0}, \\ \xi_k \in [10^{-3}, 1]\ \forall k = 1, \dots, (n_u + 1) \times (n_v + 1). \end{cases} \tag{12}$$

In problem (12), $l(\boldsymbol{\xi})$ is the virtual work of the applied loads, E^{0}_{ijkl} the standard stiffness tensor of the isotropic material, $\alpha \geq 3$ a suitable parameter that aims at penalising all the meaningless densities between 0 and 1 and $\mathbf{g}(\xi)$ is the vector collecting the technological constraints related to the considered AM process. The FEM discretised version of problem (12) is

$$\min_{\boldsymbol{\xi}} \{\mathbf{F}\} \cdot \{\mathbf{U}_{\mathrm{FEM}}\} = \min_{\boldsymbol{\xi}} c(\rho(\boldsymbol{\xi})),$$
$$\text{subject to}$$
$$\begin{cases} \left(\displaystyle\sum_{e=1}^{N_e} {\rho_e}^{\alpha} [\mathbf{K_e}]\right) = [\mathbf{K}], \\ \dfrac{V(\rho_e)}{V_{tot}} = \dfrac{\sum_{e=1}^{N_e} \rho_e}{e_x e_y} = f, \\ \{\mathbf{g}(\boldsymbol{\xi})\} \leq \{\mathbf{0}\}, \\ \xi_k \in [10^{-3}, 1]\ \forall k = 1, \dots, (n_u + 1) \times (n_v + 1). \end{cases} \tag{13}$$

In equation (13), $c(\rho)$ is the compliance of the structure and ρ_e is the value of the pseudo-density for the generic element,

$$\rho_e = \rho(u_e, v_e) = \rho\left(\frac{x_e}{w}, \frac{y_e}{h}\right), \tag{14}$$

where (x_e, y_e) are the Cartesian coordinates of the element centroid, whilst $[\mathbf{K}]$ is the global stiffness matrix obtained by the single element stiffness matrix $[\mathbf{K_e}]$ and e_x and e_y are the number of mesh divisions along x and y axes, respectively.

The SIMP approach revisited in the NURBS mathematical framework is characterised by a given number of features which implies just as many advantages:

1. the number of design variables is unrelated to the number of elements. In the classic SIMP approach, each element introduces a new design variable. In the NURBS framework, the accuracy of the topology description is characterised solely by the number of points of the control net, i.e. $(n_u + 1) \times (n_v + 1)$;

2. the locally supported blending functions imply an implicitly defined filter zone. The size of such a filter zone is related to the dimensions of the local support of the blending functions. It should be remarked that standard TO filters create a mutual dependency area among the elements densities, i.e. the design variables. In the case of the NURBS, the inter-dependence is automatically provided between the NURBS control points, without the need of defining a filter on the mesh elements densities.

3. the NURBS formalism allows taking into account new kinds of constraints, since a mathematically well-defined description of the geometrical bounds of the optimum topology is always available during the iterations of the optimisation process.

4 Numerical Strategy

In this section a suitable numerical strategy for solving the CNLPP (13) is presented. A synthetic scheme of the numerical strategy is illustrated in Figure 1:. Only few comments are added in order to clarify the procedure.

Pre-processing: both a mesh and a NURBS parametrisation are associated to the geometrical reference domain. The boundary conditions and loads are set. The user can enable a symmetric solution (i.e. a symmetric shape of the Bspline scalar function defining the pseudo-density). At this stage the user has to set the objective function as well as the optimisation constraints for the problem at hand.

Initialisation: for a given problem usually the pseudo-density field is initialised in order to satisfy the volume constraint at the beginning of the optimisation.

Optimisation Block: it should be remarked that sensitivity analysis is not automatically activated; some problems have simple objective and constraints functions, so derivatives can be easily provided in analytical form. However, the algorithm, in its most general form, does not require the gradient provision and it can be adequate for whatever customised problem.

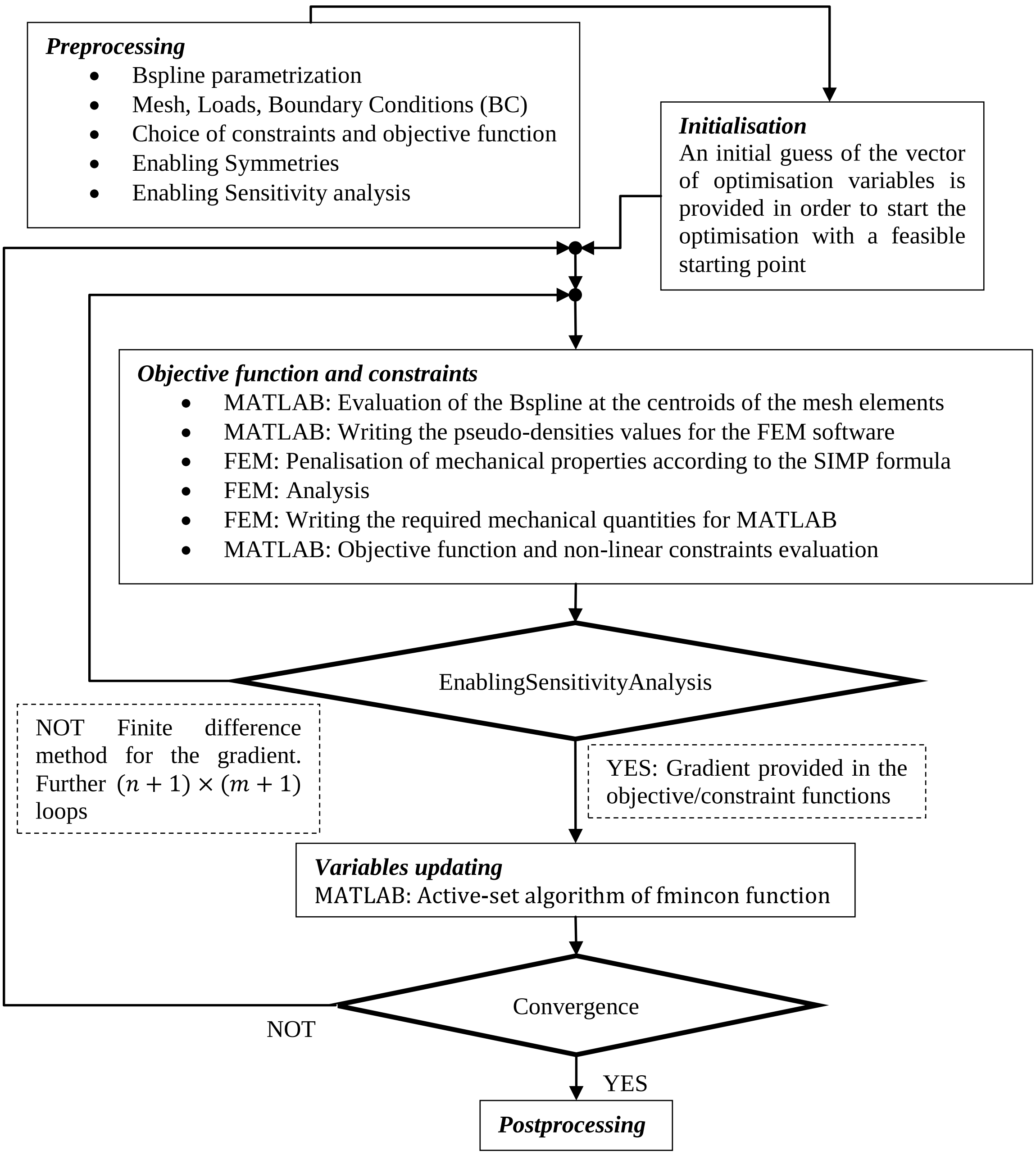


Figure 1: The numerical strategy – synthetic scheme.

5 Results

In order to prove the effectiveness of the proposed approach several benchmarks and real-world engineering problems have been analysed. However, for the sake of brevity, in this section only some meaningful results related to the "cantilever plate" benchmarkillustrated in Figure 2 arediscussed. The

Figure 2: The proposed benchmark – In-plane dimensions: w =320mm, h=200 mm. Thickness: t=2 mm. Material: E=72000 MPa, $\nu = 0.33$. Load: P=1000 N.

results including a technological constraint on the local radius of curvature (together with other meaningful benchmarks) will be presented in an extended version of this manuscript.

The aim is to minimise the compliance by keeping the volume of the structure at the 40% of the starting volume. All geometrical and mechanical data are provided in the caption of Figure 2.

Figure 3 shows a typical result of the TO analysis: the pseudo-density NURBS function. The corresponding optimised structure is depicted in Figure 4 and it is obtained by means of the intersection of the aforementioned NURBS with a suitable cutting plane. For all the considered benchmarks, the compliance is evaluated after cutting the Bspline surface with the cutting plane and compared with the value provided by the TO algorithm at the end of the analysis. This comparison (in terms of objective function values) is considered in order to prove the consistency of the proposed method.

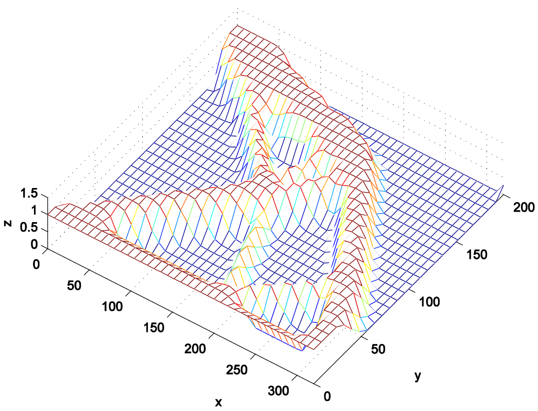


Figure 3: Example of pseudo-density described by means of a Bspline function

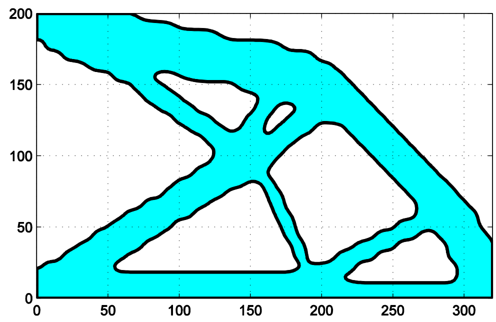


Figure 4: Optimised structure at the end of the NURBS-based TO method

The first campaign of analyses aims at investigating the effects of the filter zone dimensions on the final topology. Being the filter zone affected by the discrete parameters of the NURBS, the following analyses have been performed by changing both the NURBS degrees and the number of control points. Results are collected in Table 1 in the case of a fixed mesh of 40×25 SHELL elements with 8 nodes and 6 degrees of freedom per node.

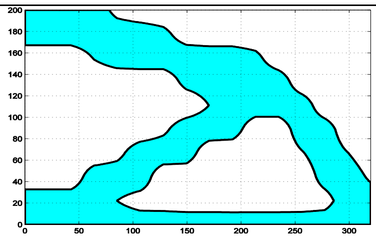
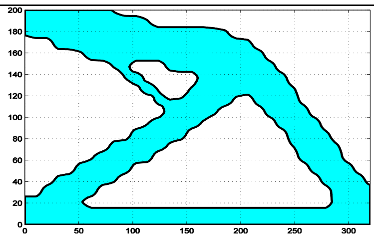
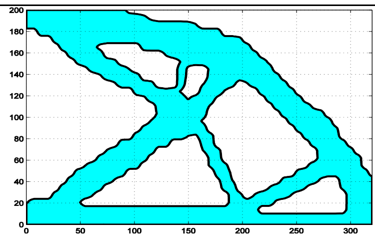
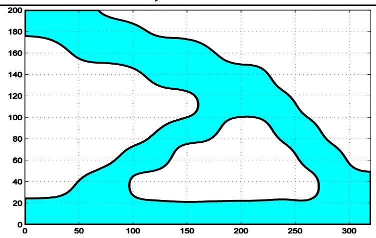
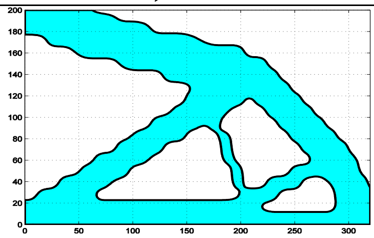
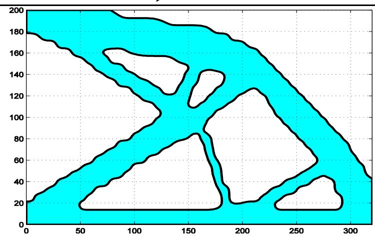
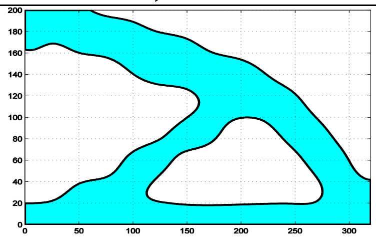
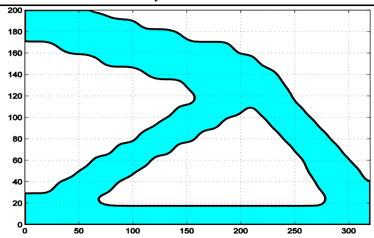
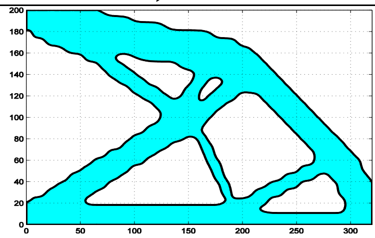
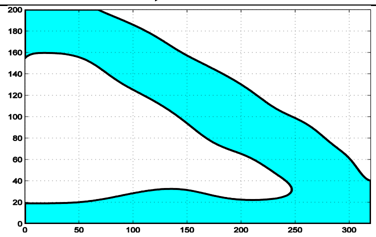
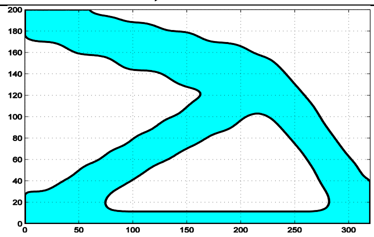
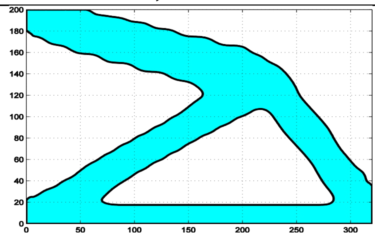
	16x10 Control Points	32x20 Control Points	48x30 Control Points
p,q=1	 Compliance=427,55 J V=0,4008Vtot	 Compliance=401,78 J V=0,3999Vtot	 Compliance=403,45 J V=0,4017Vtot
p,q=2	 Compliance=414,35 J V=0,4015Vtot	 Compliance=387,91 J V=0,4217Vtot	 Compliance=398,7261 J V=0,4004Vtot
p,q=3	 Compliance=422,78 J V=0,4010Vtot	 Compliance=398,74 J V=0,4027Vtot	 Compliance=393,48 J V=0,4016Vtot
p,q=6	 Compliance=519,49 J V=0,4009Vtot	 Compliance=407,95 J V=0,4022Vtot	 Compliance=397,94 J V=0,3999Vtot

Table 1: Sensitivity of the solution to the filter dimensions

The dimensions of the filter increase when the degrees increase or when the number of control points decreases. So, evident changes in resulting topologies occur: when the number of control points increases the final optimum topology has better quality (together with better performances) and thinner features (i.e. thin branches) appear. Conversely, increasing the degrees implies an inhibition of such features. Hence, it is evident that the dimension of the filter zone affects the minimum member size that can be expected from the topology optimisation. It should be also highlighted that, if objective function values are compared, only the solution $p, q = 6$ with 16×10 control points is significantly far from the other solutions: it can be explained by the fact that the filter dimensions are too big and the zone of interdependence among elements is too extended. So, the algorithm tends to converge on a pseudo-optimal solution. However, increasing too much the number of control points or decreasing the degree of the blending functions does not imply a more efficient solution (in terms of both objective and constraint functions).

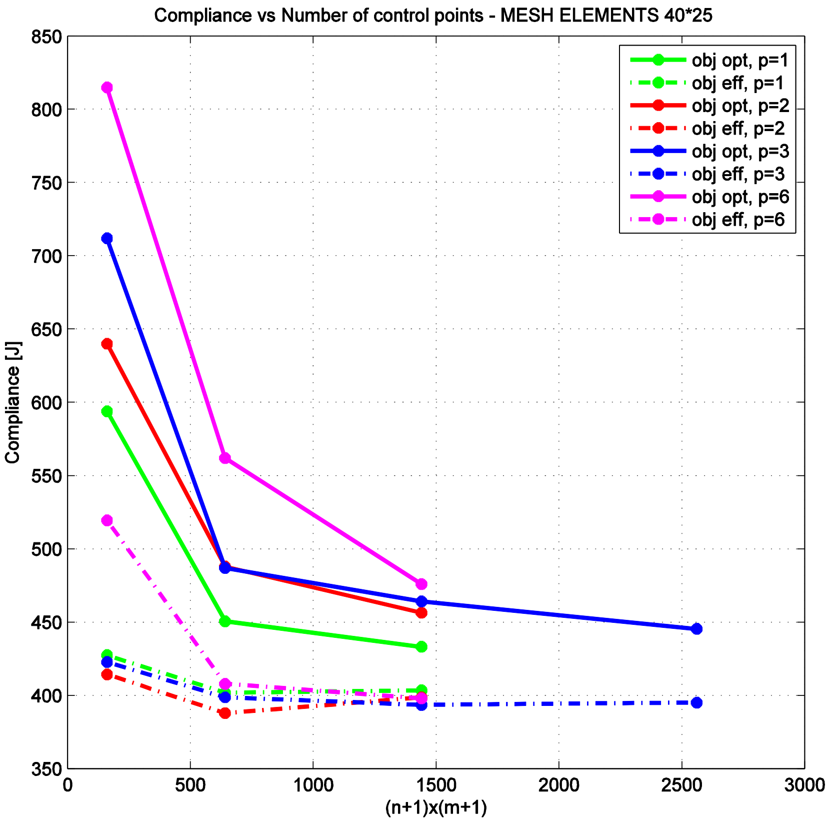


Figure 5: Objective function vs number of control points for a 40x25 mesh elements

Furthermore, too small filter dimensions lead to misleading results. When the filter dimensions are lower than or equal to those of the elements, the checkerboard effect appears also in the framework of the NURBS-based SIMP approach.

Concerning the volume equality constraint, it is strictly met in the examined configurations (after performing the geometrical reconstruction of the optimum topology). Indeed this is a strong advantage of the NURBS-based SIMP approach: when the pseudo-density field is described through a NURBS scalar function, it is automatically compatible with any standard format of data exchange (IGS, STEP, etc.) and the optimum topology can be easily transferred from the FE code to a CAD software without the need of any curve/surface fitting phase. Conversely, in the framework of the classical SIMP approach (where the volume constraint is met only in the element-discretised domain) there is not any ad-hoc rule to retrieve the boundary of the optimum topology by rigorously satisfying the volume constraint during CAD rebuilding phase (often the optimum topology is described through the positions of the elements nodes at the end of the analysis and requires complex surface and/or curve fitting operations which lead to a considerable increase of the volume of the final topology).

Moreover, Figure 5 shows the trends of the compliance versus the number of control points for several values of the surface degrees. In this figure, the objective function at the end of the optimisation is called “obj opt” and it is the nominal compliance of the structure evaluated on the whole domain D with a mapped mesh (it is represented with a continuous line). The effective compliance of the rebuilt structure (i.e. the compliance values reported in Table 1) is marked with dashed lines.

From an accurate analysis of results provided in Table 1 and Figure 5 two basic facts can be deduced:

for each analysis the effective compliance is always smaller than the nominal one. This means that the proposed methodology is conservative (in terms of the strain energy of the structure);
when the number of control points reaches a threshold value (when the number of control point is about the 75% of the mesh elements) it has no more influence on the value of the compliance. This means that even the user chooses of increasing the number of control points beyond this threshold there is almost any influence on the values of the objective/constraint functions. This fact also proves that the number of design variables is unrelated to the mesh size and, if the aforementioned constraints on the filter dimensions are met, the designer is free to choice the best compromise between computational time and accuracy in the description of the involved physical phenomena.

6 Conclusions

This study aims at proving the possibility of enhancing the classic SIMP approach in the context of the NURBS formalism for 2D structures. The main effects of such a choice have been investigated and the main results can be summarised as follows.

1. The NURBS representation of the pseudo-density introduces an implicitly defined filter zone that should be properly sized by means of the NURBS discrete parameters in order to avoid numerical artefacts or premature convergence on pseudo-optimal solutions.
2. If the dimensions of the filter are big enough (i.e. superior to the mesh characteristic dimension) in order to prevent the checkerboard effect, there is a substantial independence of the resulting objective function from the number of the NURBS control points. Therefore, increasing the number of design variables beyond to a given threshold value (which depends upon the problem at hand) does not affect the result in terms of objective and constraint functions.
3. The final rebuilt structure (i.e. the CAD geometrical representation of the optimum topology) exhibits conservative and consistent properties in terms of both the objective function and the volume constraint: for the considered examples the CAD representation of the optimum solution has always the same (or a lower) objective function value (when compared to that provided by the TO algorithm) and exactly meets the volume constraint.
4. Using the NURBS allows for precisely describing the structure boundaries, so unconventional constraints related to the AM technology can be imposed. In this paper a constraint on the radius of curvature has been successfully included in the TO.

This work opens several perspectives: first of all, some constraint, typical of the AM technology, can be included in the TO. In this sense, the most important constraints to be taken into account are the minimum length scale size and the volume of support. The first constraint should be imposed on the true boundary of the structure and not on the mesh elements. Therefore, the minimum length constraint would exactly correspond to the actual minimum printable feature size. Concerning the latter constraint, it can be stated that the most efficient way to deal with support structures could be a minimisation of their volume rather than avoiding their presence on the final product. Finally, the most challenging perspective is to develop the NURBS-based SIMP approach in the most general 3D case.

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