Thermal buckling and post-buckling of laminated composite plates with temperature dependent properties by an asymptotic numerical method

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ABSTRACT

This paper presents a methodological approach based on the homotopy and perturbation methods for thermal buckling and post-buckling analyses of the anisotropic laminated plates with temperature dependent properties. A power law distribution in terms of temperature is used and the structure is subjected to a uniform temperature variation. A mathematical formulation that may account for various temperature dependent models is elaborated. Power series expansions of the displacement and the temperature are developed and the finite element method is used for numerical solutions. The critical buckling load and the post-buckling equilibrium path of plates under thermal loading are investigated. The effects of temperature dependent properties, structure geometry and boundary conditions on the thermal buckling and post-buckling behaviours are evaluated through parametric studies.

Keywords : thermal buckling and postbuckling/finite element method/homotopy method/laminated composite plate/temperature-dependent properties.

Nomenclature:

A_{ij}: *component of extensional stiffness of laminate*

b: width of the plate

B_{ij}:*component of bending-extensional stiffness of laminate*

D_{ij}: *bending stiffness of laminate*

E_1 , E_2 *: Young's modulus in the fiber and transverse to the fiber direction*

G_{12}: *Shear modulus*

h: total thickness of the plate

h_k: *thickness of the* k^{th} *core layer*

L: length of the plate

Q_{ij} *: material stiffness coefficients*

U: vector of the nodal displacement of the plate

T: temperature

T_0: *free stress temperature*

x, y and z : coordinates on the middle surface of a plate

θ*: Lamination angle*

ν: *Poisson's ratio of lamina*

α_{ij}: *coefficient of thermal expansion*

Δ T : temperature rise (ΔT =T-T0)

1. Introduction

Thin-walled structures such as beams and plates can become unstable at a relatively low temperature and buckle in the elastic region. Composite plate structures are often subjected to elevated temperatures. In such circumstances, high thermally induced compressive stresses will be developed in the constrained plates and consequently will lead to buckling.

Since they retain considerable post-buckling strength beyond the thermal buckling load, it is quite advantageous to make use of the post-buckling characteristics in practical design. Most of the investigations on the subject of thermal post-buckling have been devoted to thin structures, [1,2] in which the elastic and thermal properties are considered independent of temperature.

Most of the investigations on the subject of thermal buckling do not involve the effect of temperature-dependent material properties. However, elastic and thermal properties are known to vary with the change of temperature. The temperature-dependent material effect lowered the critical buckling temperatures and increased the post-buckling deflections. The investigation of thermal post-buckling of laminated composites, considering material degradation using finite element methods, and the static buckling of composite and sandwich plates under thermal loads using layer wise plate theories are presented in [3,4,5].

The aim of this paper is the development of a path following algorithm to calculate the critical thermal buckling and the post-buckling equilibrium path using an asymptotic numerical method [6-7].Temperature-dependent elastic and thermal properties are considered. A power law distribution in terms of temperature is used and the structure is subjected to a uniform temperature change. The effects of temperature dependent properties, structure geometry and boundary conditions on the thermal buckling and post-buckling behaviours are evaluated through parametric studies. A mathematical formulation that may account for various temperature dependent models is proposed. Power series expansions of the displacement and the temperature are developed and the finite element method is used for numerical solutions.

2. Finite element formulation

A general composite laminated plate with constant thickness h is considered here. The rectangular coordinates x, y and z are taken on the middle surface of a plate as shown in (Fig. 1). The kinematic plate model is based on the first-order shear deformation theory.

The present work focuses on thermal buckling and post-buckling behaviors of anisotropic laminated composite plate with temperature-dependent material properties. The material properties, such as Young's moduli E_i and thermal expansion coefficients α_i, can be expressed as a linear function of temperature 'T' as:

$$E_1(T) = E_{10}(1 + E_{11}T)\,, E_2(T) = E_{20}(1 + E_{21}T)\,, G_{12}(T) = G_{120}(1 + G_{121}T)\,,$$

$$\alpha_1(T) = \alpha_{10}(1 + \alpha_{11}T)\,, \alpha_2(T) = \alpha_{20}(1 + \alpha_{21}T), \tag{1}$$

The Poisson coefficient ν is considered in this study to be temperature independent. The stress-strain relation of the plate, subjected to temperature rise ΔT, is given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_y - \alpha_y \Delta T \\ \gamma_{xy} - \alpha_{xy} \Delta T \end{Bmatrix} \tag{2}$$

in which $\left[\overline{Q}_{ij}(\mathrm{T})\right]$ are the material stiffness coefficients that are temperature dependent.

The membrane stress and the bending moment resultants, {N} and {M}; and their relation to the membrane strains are given by:

$$\{N\}=\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}dz\ ,\quad \{M\}=\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}zdz \qquad \text{(3-a)}$$

$$\begin{Bmatrix} N \\ M \end{Bmatrix}=\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \varepsilon^{l} \\ \kappa \end{Bmatrix}-\begin{Bmatrix} N^{T} \\ M^{T} \end{Bmatrix} \qquad \text{(3-b)}$$

where the temperature dependent matrices A, B and D can be expressed as:

$$\left[A_{ij}(T), B_{ij}(T), D_{ij}(T)\right]=\sum_m \int_{z_m}^{z_{m=1}} Q_{ij}(T)(1,z,z^2)dz \qquad (4)$$

The normalized thermal force and thermal moment resultants are defined as

$$\{N^T\}=\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix}=\sum_{k=1}^{n}\int_{h_k}^{h_{k+1}} Q_{ij}^{(k)}(T)\begin{Bmatrix} \alpha_1(T) \\ \alpha_2(T) \\ \alpha_{12}(T) \end{Bmatrix}\Delta T dz \qquad \text{(5-a)}$$

$$\{M^T\}=\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix}=\sum_{k=1}^{n}\int_{h_k}^{h_{k+1}} Q_{ij}^{(k)}(T)\begin{Bmatrix} \alpha_1(T) \\ \alpha_2(T) \\ \alpha_{12}(T) \end{Bmatrix}z\Delta T dz \qquad \text{(5-b)}$$

2.1. Thermal buckling:

Based on the previous assumptions and using the finite element method, the thermal buckling is governed by the following nonlinear eigenvalue problem:

$$[K_e(T)]\{U\}=T\left(\left[K_{g0}\right]-\left[K_{e1}\right]\right)\{U\}+T^2\left[K_{g01}\right]\{U\}+T^3\left[K_{g11}\right]\{U\} \qquad (6)$$

where [K_e(T)] and [K_g(T)] are the resulting temperature dependent stiffness and geometric matrices.

As this nonlinear eigenvalue problem can not be solved with classical codes, the homotopy technique [7] is used for numerical solutions. This allows us to introduce artificially a parameter ‘a’ in Eq. (6) as

$$[K_{e0}]\{U\} = T.\left([K_{g0}] - [K_{e1}]\right)\{U\} + \mathbf{a}\left(T^2[K_{g01}]\{U\} + T^3[K_{g11}]\{U\}\right)\ 0 \le a \le 1 \tag{7}$$

A numerical procedure is elaborated herein to compute the critical buckling temperatures and the associated eigenmodes. One can easily note that the initial problem (6) corresponds to a=1.

2.2. Thermal post-buckling behaviour

The critical buckling and the corresponding temperature dependent eigenmodes are numerically computed for beams and plates. After the buckling prediction, the nonlinear equilibrium equation for the symmetrically laminated beams and plates under uniform temperature rise may be expressed as:

$$\langle L_e(T)U, \delta U\rangle - T\langle L_g(T)U, \delta U\rangle + \langle Q(U,U)(T), \delta U\rangle = 0 \tag{8}$$

in which L_e, L_g are linear operators and Q is a quadratic one that are temperature dependent. These operators are built following the same procedure elaborated in [6]. The corresponding matrix operators, L_e, L_g, are the linear stiffness and the geometric stiffness matrices, respectively.

2.2.1. Asymptotic numerical method

It should be noted that the variational problem (8) is hardly nonlinear with respect to temperature T. A well-adapted numerical procedure is thus required for numerical solution. For this aim an asymptotic numerical algorithm is elaborated herein to solve the resulting nonlinear thermal problem with a reasonable computational cost. This algorithm combines the perturbation technique and the finite element.

The displacement and temperature are expanded into power series around a starting solution (U_0,T_0) in the following form:

$$\begin{cases} T = T_0 + aT_1 + a^2T_2 + \cdots + a^nT_n \\ T^2 = p = p_0 + ap_1 + a^2p_2 + \cdots + a^np_n \\ T^3 = c = c_0 + ac_1 + a^2c_2 + \cdots + a^nc_n \\ U = U_0 + aU_1 + a^2U_2 + \cdots + a^nU_n \end{cases} \quad (9)$$

These power series expansions are used to compute the path post-buckling behavior with respect to temperature for various structures geometries and boundary conditions. For isotropic materials with the Poisson ratio temperature independent, the following variational formulations are resulted.

$$\langle L_e(T)U, \delta U\rangle = \langle L_{e0}(E_{i0})U, \delta U\rangle + \mathrm{T}\langle L_{e1}(E_{i1})U, \delta U\rangle \quad \text{(10-a)}$$

$$\langle L_g(T)U, \delta U\rangle = \langle L_{g0}(E_{i0}, \alpha_{i0})U, \delta U\rangle + T\big(\langle L_{g01}(E_{i1}, \alpha_{i0})U, \delta U\rangle + \langle L_{g10}(E_{i0}, \alpha_{i1})U, \delta U\rangle\big) + T^2\langle L_{g11}(E_{i1}, \alpha_{i1})U, \delta U\rangle \quad \text{(10-b)}$$

$$\langle Q(U,U), \delta U\rangle = \langle Q_0((E_{i0}))(U,U), \delta U\rangle + T\langle Q_1((E_{i1}))(U,U), \delta U\rangle \quad \text{(10-c)}$$

Using homotopy procedure and the decomposition (10-a-b-c), the finite element form of the thermal post-buckling leads to the followingelemental matrixequation:

$$\langle L_{e0}U, \delta U\rangle + \langle Q_0(U,U), \delta U\rangle + T\big(\langle L_{e1}U, \delta U\rangle - \langle L_{g0}U, \delta U\rangle + \langle Q_1(U,U), \delta U\rangle\big) + \mathbf{a}\big(T^2\langle L_{g01}U, \delta U\rangle + T^3\langle L_{g11}U, \delta U\rangle\big) = 0 \quad (11)$$

After insertion of series (9) into Eq. (11), one gets the following recurrent linear problems that come from the identifications of the like powers of a:

Order 0:

$$\langle L_{e0}U_0, \delta U\rangle + \langle Q_0(U_0, U_0), \delta U\rangle = T_0\big(\langle L_{g0}U_0, \delta U\rangle - \langle L_{e1}U_0, \delta U\rangle - \langle Q_1(U_0, U_0), \delta U\rangle\big) \quad (12\text{-a})$$

Order 1:

$$\langle L_T U_1, \delta U\rangle = T_1\big(\langle L_{g0}U_0, \delta U\rangle - \langle L_{e1}U_0, \delta U\rangle\big) + p_0\langle L_{g01}U_0, \delta U\rangle + c_0\langle L_{g11}U_0, \delta U\rangle \quad (12\text{-b})$$

Order k (k≥2): $\langle L_T U_k, \delta U\rangle = T_k\{F_1\} +$ (12-c)

with: - $\{F_k\} = \sum_{i=1}^{k-1} T_i\langle L_{g0}U_{k-i}, \delta U\rangle - \sum_{i=1}^{k-1} T_i\langle L_{e1}U_{k-i}, \delta U\rangle + \sum_{i=0}^{k-1} p_i\langle L_{g10}U_{k-i-1}, \delta U\rangle + \sum_{i=0}^{k-1} c_i\langle L_{g11}U_{k-i-1}, \delta U\rangle - \sum_{i=1}^{k-1}\langle Q_0(U_i, U_{k-i}), \delta U\rangle - \sum_{i=1}^{k-1} T_i \sum_{j=1}^{k-i-1}\langle Q_1(U_j, U_{k-j-i}), \delta U\rangle$

- $\{F_1\} = \big(\langle L_{g0}U_0, \delta U\rangle - \langle L_{e1}U_0, \delta U\rangle\big)$
- $\langle L_T U_i, \delta U\rangle = \langle L_{e0}U_i, \delta U\rangle + 2\langle Q_0(U_0, U_i), \delta U\rangle - T_0\big(\langle L_{g0}U_i, \delta U\rangle - \langle L_{e1}U_i, \delta U\rangle\big) \quad i \geq 1$

in which

- $\begin{cases} p_0 = T_0^2 \\ p_1 = 2T_0T_1 \\ p_k = 2T_0T_k + \sum_{i=1}^{k-1} T_iT_{k-i} = 2T_0T_1 + D_k, for\ k \geq 2 \end{cases}$

- $\begin{cases} c_0 = T_0^3 \\ c_1 = 3T_0^2T_1 \\ c_k = 3T_0^2T_k + D_kT_0 + \sum_{i=1}^{k-1} p_iT_{k-i} = 3T_0^2T_1 + A_k, for\ k \geq 2 \end{cases}$

Remember that the left hand sides of the problems Eqs. (12-a,b,c) have the same matrix. Thus, only one matrix inversion is needed for all vectors U_j. This methodological approach allows computing the power series coefficients T_j and U_j at any required order. Based on this approach, the thermal post-buckling equilibrium path can be easily investigated for plates with various shapes and temperature dependent models

3. Numerical results

In this section, firstly, the formulation and the method of solution are validated by comparing the results with those available in the literature. Then, the resultsforthermal buckling and post-bucklinganalysisof laminated composite plates with temperature-dependent material properties are presented.

The thermal buckling and post-buckling with properties dependent temperature of a plate is performed to demonstrate the accuracy and validity of the present numerical method. The plate is simply supported with L=b and L/h=100.

All the material parameters are taken from Ref. [3].

$$E_1(T) = E_{10}(1 + E_{11}T)\,, E_2(T) = E_{20}(1 + E_{21}T)\,, G_{12}(T) = G_{120}(1 + G_{121}T)$$

$\alpha_1(T) = \alpha_{10}(1 + \alpha_{11}T)\,, \alpha_2(T) = \alpha_{20}(1 + \alpha_{21}T), E_{10}/E_{20} = 40,$

$G_{120}/E_{20} = 0{,}5, \alpha_{10} = 10^{-6}°\mathrm{C}^{-1}\,, \alpha_{20} = 10^{-5}°\mathrm{C}^{-1},\ \nu_{12} = 0{,}25,$

Materiel1:$E_{11} = -0{,}5.10^{-4}\ , -0{,}1.10^{-3}\ , -0{,}2.10^{-3}°\mathrm{C}^{-1}$

$$E_{21} = G_{121} = \alpha_{11} = \alpha_{21} = 0$$

Materiel2:$E_{11} = -0{,}5.10^{-3}°\mathrm{C}^{-1}, E_{21} = G_{121} = -0{,}2.10^{-3}\ °\mathrm{C}^{-1}$,

$\alpha_{11} = \alpha_{21} = 0{,}5.10^{-3}°\mathrm{C}^{-1}$, We validate our methodology for simply-supported$(\pm 45_6)_T$laminate square plates. The temperature change versus the maximum deflection of the plate with the variation of $E_{11}\,(material\,1)$ are shown in (Table 1). At the same maximum deflection of the plate, the temperature change decreases as the absolute value ofE_{11}, increases for the simply-supported boundary plate. The results of Chen et al. [3] and Shen [6] by using the Finite Element method and Reddy's higher-order shear deformation theory are also displayed for direct comparison. From Table 1, an excellent agreement is observed.

Figure 2 shows the effect of temperature-dependent properties on the thermal post-buckling behavior of simply-supportedplate. The material properties effect is demonstrated for IDT (E_{11}=0), materiel 1 and material 2.

After comparing the present solutions with those of existing in literature, it is observed that the thermal buckling strength has been reduced significantly when the temperature-dependent properties are taken into consideration.

4. Conclusion

An asymptotic numerical method is employedto compute the thermal buckling and post-buckling analysis of laminated plates. The material properties were assumed to be temperature-dependent. Plates with different boundary conditions and temperature dependent models were considered. After obtaining the thermal buckling bifurcation point, the nonlinear equilibrium equations were employed to get the post-buckling configurations. The influence of material property with respect to temperature considerably affects the thermalbuckling temperature and post-buckling path.

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Figures :

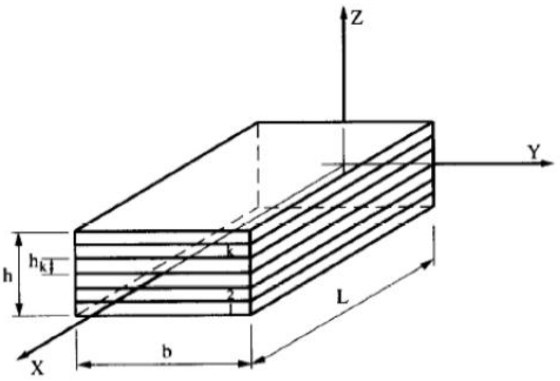

Figure 1: The geometry of laminate plate

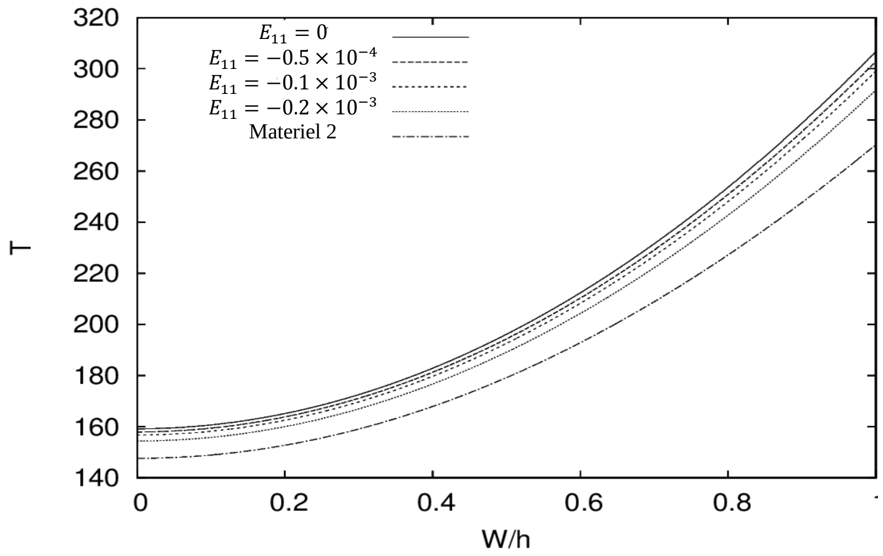

Figure 2: Influence of temperature dependency of material on post-buckling temperature of simply supported squared plate L/b=1 , L/h=100.

Tables :

Table 1: Comparison of thermal postbuckling loads for $(\pm45_6)_T$ laminated square thin plates subjected to an uniform temperature rise

	$E_{11} = 0$			$E_{11} = -0.5 \times 10^{-4}$			$E_{11} = -0.1 \times 10^{-3}$			$E_{11} = -0.2 \times 10^{-3}$		
W_c/h	present method	[6]	[3]	present method	[6]	[3]	present method	[6]	[3]	present method	[6]	[3]
0	159.19	158.18	159.64	159.43	158.14	159.50	158.20	158.08	159.34	155.80	157.85	158.92
0.1	160.69	159.47	160.91	160.82	159.43	160.78	159.59	159.38	160.62	157.13	159.16	160.21
0.2	165.14	163.34	164.72	165.31	163.31	164.60	164.01	163.26	164.46	161.41	163.04	164.07
0.3	172.53	169.80	171.09	172.88	169.76	170.98	171.47	169.71	170.86	168.63	169.47	170.52
0.4	182.87	178.85	180.00	183.55	178.79	179.92	181.96	178.71	179.84	178.78	178.41	179.58