Modelling transitional falling liquid films

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Résumé :

Les conditions opératoires des échangeurs à films ruisselants correspondent à l'intervalle 100 à 300 du nombre de Reynolds pour lequel le développement d'ondes de surface peut s'accompagner de la présence de turbulence. Nous proposons dans cette étude de formuler des équations simplifiées semblales aux équations de Saint-Venant et consistantes avec le développment "ondes longues" à l'aide de la méthode aux résidus pondérés. Le modèle se présente sous la forme d'équations d'évolution pour l'épaissseur h et le débit q. Chaque coefficient du modèle est fonction du nombre de Reynolds local $|q(x,t)|/\nu$. Les comparaisonsdes vitesses et des profils des ondes propagatives 'à rouleaux' avec les expériences de Brock [3] sont assez bonnes.

Abstract

We consider a wavy liquid film flow, which is partly laminar and partly turbulent. Such a situation may occur for the large-amplitude solitary wave regime which develops at Reynolds numbers in the range 100 to 300, which corresponds precisely to the operational conditions of current chemical devices. We propose to model the onset of turbulence by a simple eddy-viscosity formulation using the Van Driest mixing length formula. A simplified two equation model is then derived within the framework of the weighted residual method in terms of the local flow rate $q(x,t)$ and the film thickness $h(x,t)$. The model is consistent with the asymptotic long-wave expansion, which guarantees that instability thresholds are correctly captured, and accounts for surface tension and elongational viscosity. Each coefficient of the model is a function of the local Reynolds number, $|q(x,t)|/\nu$ and has been tabulated numerically. Preliminary comparisons to the experimental data by Brock [3] for roll waves in the full turbulent regime prove to be satisfactory. In particular, the values of the coefficient of the Chezy law proposed to model the wall shear stress using Brock's experiments are recovered without adjusting parameters.

Mots clefs : falling films, instabilities, transition to turbulence, wall induced turbulence

Introduction

The design of compact and efficient heat and mass exchangers is a key issue in chemical engineering processes. It requires the intensification of heat and mass transfer at the local scale, which is generally obtained by increasing the contact area between the fluids and generating mixing. In the case of two-phase flow processes, such as evaporators or absorbers, falling film plate exchanger are solutions of interest whenever the pressure drop in the gas phase is a key issue, a situation that commonly arises when low-temperature heat sources are involved, for instance in the design of recovery systems to limit energy waste.

Usual operating conditions of falling-film plate exchangers correspond to Reynolds numbers lying in the range 50 to 1000, in which case a wavy regime characterized by large-amplitude long waves in interaction develops. These waves significantly increase the heat and mass transfer across the film by thinning the film, which reduces the resistance to transfer, and by providing an efficient mixing mechanism via wave merging. For these operating conditions, Ishiga et al. [5] noticed a transition from a laminar wavy state to a wall-induced turbulent state. This transition occurs at far lower Reynolds numbers than what is typically observed in pipe flows and significantly affects heat transfer across the film. As a consequence, a reliable modelling of falling film flows in the typical conditions encountered in industrial applications requires to account for the onset of wall-induced turbulence, which is probably localized under the crests of the most prominent waves as suggested by the experiments of Adomeit and Renz [4] and the numerical study of Dietze et al. [1].

In this paper, we outline the derivation of a consistent set of averaged equations across the film layer, which accounts for the presence of wall-induced turbulence within the framework of a zero-equation closure.

Formulation

We consider a falling liquid film on an inclined plane under the action of gravity. The flow is assumed incompressible, Newtonian and to remain two-dimensional (spanwise invariant). The streamwise coordinate is denoted by x, the cross-stream coordinate is y. The thermophysical properties, density ρ, dynamic and kinematic viscosities, μ and $\nu = \mu/\rho$, and surface tension σ are constant. θ refers to the inclination angle of the plane. We introduce different length scales for the coordinates in order to account for the long-wave nature of the instability of film flows. The pressure p, velocity field (u, v), film thickness h and stresses τ are made dimensionless by introducing the typical length of the wave, L, the typical thickness h_0 and velocity scale $\langle u_0 \rangle$:

$$\tilde{u} = \frac{u}{\langle u_0 \rangle}; \quad \tilde{v} = \frac{v}{\varepsilon \langle u_0 \rangle}; \quad \tilde{x} = \frac{x}{L}; \quad \tilde{y} = \frac{y}{h_0}; \quad \tilde{t} = \frac{t}{L}\langle u_0 \rangle;$$
$$\tilde{h} = \frac{h}{h_0}; \quad \tilde{p} = \frac{p}{\rho g h_0}; \quad \tilde{\tau}_{xx} = \frac{L\tau_{xx}}{\langle u_0 \rangle \mu}; \quad \tilde{\tau}_{yy} = \frac{L\tau_{yy}}{\langle u_0 \rangle \mu}; \quad \tilde{\tau}_{xy} = \frac{h_0\tau_{xy}}{\langle u_0 \rangle \mu} \tag{1}$$

where $\varepsilon = h_0/L \ll 1$ is the film parameter. The governing equations thus read

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0\,, \tag{2a}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}^2}{\partial \tilde{x}} + \frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{y}} = \frac{\lambda}{\varepsilon Re} - \frac{1}{F^2}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\varepsilon}{Re}\frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} + \frac{1}{\varepsilon Re}\frac{\partial \tilde{\tau}_{xy}}{\partial \tilde{y}}\,, \tag{2b}$$

$$\varepsilon^2 F^2 \left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \frac{\partial \tilde{u}\tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{v}^2}{\partial \tilde{y}} \right) = -\cos\theta - \frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\varepsilon Fr^2}{Re}\frac{\partial \tilde{\tau}_{yy}}{\partial \tilde{y}} + \frac{\varepsilon Fr^2}{Re}\frac{\partial \tilde{\tau}_{xy}}{\partial \tilde{x}}\,, \tag{2c}$$

where the shear stresses are written with the Prandtl's classical mixing length hypothesis

$$\tilde{\tau}_{xy} = [1 + Re l^2(\bar{y}, h, q)\dot{\gamma}] \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} + \varepsilon^2 \frac{\partial \tilde{v}}{\partial \tilde{x}} \right), \qquad \tilde{\tau}_{xx} = -\tilde{\tau}_{yy} = 2(1 + Re l^2(\bar{y}, h, q)\dot{\gamma}) \frac{\partial \tilde{u}}{\partial \tilde{x}}, \tag{2d}$$

$$\dot{\gamma} = \sqrt{(\partial_y u + \varepsilon^2 \partial_x v)^2 + 4\varepsilon^2 (\partial_x u)^2}\,. \tag{2e}$$

This set of equations is completed by the boundary conditions. No slip at the wall $y = 0$ and kinematic boundary condition at the free surface $y = h$

$$\tilde{v}(\tilde{y} = 0) = 0\,, \qquad \tilde{u}(\tilde{y} = 0) = 0\,, \qquad \frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{u}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = \tilde{v}(\tilde{h})\,, \tag{2f}$$

and the continuity of the stresses at the free surface assuming a passive atmosphere (constant pressure and negligible shear stresses).

$$\tilde{\tau}_{xy}(\tilde{h}) + \left[\frac{\varepsilon Re}{F^2} \tilde{p}(\tilde{h}) - \varepsilon^2 \tilde{\tau}_{xx}(\tilde{h}) \right] \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{\varepsilon^3 Re}{We} \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \left[1 + \varepsilon^2 \left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \right]^{-\frac{3}{2}} = 0 \tag{2g}$$

$$\frac{Re}{Fr^2} \tilde{p}(\tilde{h}) + \varepsilon \frac{\partial \tilde{h}}{\partial \tilde{x}} \tilde{\tau}_{xy}(\tilde{h}) - \varepsilon \tilde{\tau}_{yy}(\tilde{h}) + \frac{\varepsilon^2 Re}{We} \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \left[1 + \varepsilon^2 \left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \right]^{-\frac{3}{2}} = 0\,. \tag{2h}$$

Finally, the following set of dimensionless groups have been defined, namely the Reynolds number

$$Re = \frac{h_0 \langle u_0 \rangle}{\nu}\,, \tag{3a}$$

the Froude number,

$$F = \frac{\langle u_0 \rangle}{\sqrt{g h_0}} \tag{3b}$$

and the Weber number.

$$We = \frac{\rho h_0 \langle u_0 \rangle^2}{\gamma} \tag{3c}$$

For convenience, we introduce the combinations

$$\lambda = \frac{Re \sin \theta}{F^2}\,, \qquad \kappa = \frac{\varepsilon^2 F^2}{We}\,. \tag{3d}$$

and next assume dominant surface tension effects i.e. $\kappa = O(1)$.

Let us underline that the proposed modelling of the turbulence within the film is the crudest one, one may think of. We simply consider the turbulence to be generated by the shear at the wall. We do not consider the onset of turbulence at the free surface of the film that might occur due to wave breaking phenomenon. We also oversimplify the distribution of turbulence within the film by considering it to be a function of the sole distance to the wall and the wall shear stress. We thus assume the evolution of turbulence to be sufficiently rapid to adjust to the deformation of the free surface (geometry) and local flow conditions (hydrodynamics) induced by the wavy regime of the film. Therefore the development that is proposed below must be considered as a first step forward to model the film hydrodynamics including both laminar and turbulent regions.

The long-wave assumption introduces an ordering in ε of all terms present in the basic set of equations, which enables to drastically simplify the problem at hand by dropping out small terms in comparison to the most prominent ones. Neglecting the terms of $O(\varepsilon^2)$ and higher we get from (2c),

$$\frac{\partial p}{\partial y} = -\cos\theta + \frac{\varepsilon F^2}{Re}\frac{\partial \tau_{yy}}{\partial y} + \frac{\varepsilon F^2}{Re}\frac{\partial \tau_{xy}}{\partial x} \tag{4}$$

where the tilde have been omitted for convenience. Integration over depth using the boundary condition (2h) next yields the expression of the pressure distribution

$$p|_y = -\kappa\frac{\partial^2 h}{\partial x^2} - \cos\theta(y-h) + \frac{\varepsilon F^2}{Re}\tau_{yy} + \frac{\varepsilon F^2}{Re}\frac{\partial}{\partial x}\int_h^y \tau_{xy}dy \tag{5}$$

Substitution in (2b) provides a truncated momentum balance which is similar to the Prandtl momentum balance in boundary-layer theory

$$\varepsilon Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{Re}{F^2}\left(\sin\theta - \varepsilon\cos\theta\frac{\partial h}{\partial x} + \varepsilon\kappa\frac{\partial^3 h}{\partial x^3}\right) + \varepsilon^2\frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy})$$
$$+\frac{\partial \tau_{xy}}{\partial y} - \varepsilon^2\frac{\partial^2}{\partial x^2}\int_h^y \tau_{xy}dy + O(\varepsilon^3) \tag{6a}$$

with the condition at the free surface

$$\tau_{xy}(h) = \varepsilon^2\frac{\partial h}{\partial x}[\tau_{xx}(h) - \tau_{yy}(h)] + O(\varepsilon^3) \tag{6b}$$

Equations (6), completed with the continuity equation (2b), are consistent up to $O(\varepsilon^2)$ with the classical long-wave expansion. They include surface tension effects and streamwise viscous terms, or elongational viscosity effects, sometimes labelled Trouton viscosity in the context of free falling liquid films. In spite of the adopted simplifications, to solve the system (6) and (2b) is still a formidable work as it still involves to track a moving free surface and thus a deformable interface. In what follows, we aim at describing the long-scale evolution of the film with a reduced set of unknowns which characterized the dynamics across an infinitesimal column of liquid at a given location x on the substrate.

The dynamics of the film is thus parameterized with the local film thickness h and the local flow rate $q = \int_0^h dy$, writing

$$u = \tilde{u}^{(0)} + \varepsilon\tilde{u}^{(1)} \qquad \text{with} \quad \tilde{u}^{(0)} = \frac{q}{h}f(\bar{y}) \quad , \quad \int_0^1 f d\bar{y} = 1 \quad \text{and} \quad \int_0^h \tilde{u}^{(1)}dy = 0 \tag{7}$$

with $\bar{y} = y/h$ the reduced cross-stream coordinate. The decomposition (7) implies that the correction $\tilde{u}^{(1)}$ does not contribute to the flow rate q, thus the last condition in (7). To be consistent, the velocity profile f must fulfill

$$\frac{d}{d\bar{y}}\left[\left(1 + |q|Re\bar{l}^2 f'\right)f'\right] = cst \equiv \lambda_f\,, \qquad f(0) = 0\,, \qquad f'(1) = 0 \quad \text{and} \quad \bar{l} = l/h\,. \tag{8}$$

Integration of the continuity equation (2a) using the free-surface kinematic condition (2f) yields the exact mass balance:

$$\partial_t h + \partial_x q = 0 \tag{9}$$

which is an evolution equation for the film thickness h. The weighted residual technique is invoked to derive an evolution equation for the flow rate q. The idea is to average the momentum balance (6) with an appropriate weight to retain consistency up to $O(\varepsilon)$. Substitution of (7) into (6) yields

$$\begin{aligned} \varepsilon Re\left(\partial_t\tilde{u}^{(0)}+\tilde{u}^{(0)}\partial_x\tilde{u}^{(0)}+\tilde{v}^{(0)}\partial_y\tilde{u}^{(0)}\right) &= \frac{Re}{Fr^2}\left[\sin\beta-\varepsilon\cos\beta\partial_x h+\varepsilon\kappa\partial_{xxx}h\right] \\ &+\partial_y\left[\left(1+Rel^2|\partial_y\tilde{u}^{(0)}|\right)\partial_y\tilde{u}^{(0)}\right]+\varepsilon\partial_y\left[\left(1+2Rel^2|\partial_y\tilde{u}^{(0)}|\right)\partial_y\tilde{u}^{(1)}\right] \\ &+\varepsilon^2\left\{Rel^2|\partial_y\tilde{u}^{(0)}|\partial_x\tilde{v}^{(0)}+4\partial_x\left[\left(1+Rel^2|\partial_y\tilde{u}^{(0)}|\right)\partial_x\tilde{u}^{(0)}\right]\right. \\ &\left.-\partial_{xx}\int_h^y\left(1+Rel^2|\partial_y\tilde{u}^{(0)}|\right)\partial_y\tilde{u}^{(0)}dy\right\}+O(\varepsilon^2)\,, \qquad (10a) \\ \tilde{u}^{(1)}|_{y=0} &= 0\,, \qquad \partial_y\tilde{u}^{(1)}|_{y=h}=O(\varepsilon^2) \qquad (10b) \end{aligned}$$

Equation (10) retains correctly the leading order contribution of the principal physical effects: inertia, gravity, surface tension, viscous drag and elongational viscosity. The loss of consistency at second order ($O(\varepsilon^2)$) corresponds to the neglect of the corrections to the inertia terms due to the deviations $\tilde{u}^{(1)}$ from the flat-film base flow. The set of equations (10) is linear with respect to $\tilde{u}^{(1)}$ and can be straightforwardly integrated to yield $\tilde{u}^{(1)}$, which would be a function of q, h and their derivatives. The gauge condition $\int_0^h \tilde{u}^{(1)}dy=0$ then provides the missing evolution equation for q, which combined with the mass balance (9), gives a closed set of reduced equation, or model, which describes the evolution of the film.

The weighted residual technique offers a useful shortcut to obtain the evolution equation for q through a careful choice of the weight function. Writing formally (10a) as $BL(\tilde{u}^{(1)})=0$, we introduce the scalar product $\langle\cdot|\cdot\rangle=\int_0^h\cdot dy$ and write the residual $\mathcal{R}=\langle BL(\tilde{u}^{(1)})|w\rangle=0$. Similarly, writing $\mathcal{L}(\tilde{u}^{(1)})=\partial_y\left[\left(1+2Rel^2|\partial_y\tilde{u}^{(0)}|\right)\partial_y\tilde{u}^{(1)}\right]$, the only term involving $\tilde{u}^{(1)}$ in the residual $\mathcal{R}$ comes from the computation of the drag

$$\langle\mathcal{L}(\tilde{u}^{(1)})|w\rangle=\langle\tilde{u}^{(1)})|\mathcal{L}^\dagger(w)\rangle \qquad (11)$$

where $\mathcal{L}^\dagger$ is the adjoint operator to $\mathcal{L}$. Two integration by parts suffice to show that $\mathcal{L}=\mathcal{L}^\dagger$ is self-adjoint. The $O(\varepsilon)$ contribution to the drag (11) can be cancelled out using the gauge condition $\langle\tilde{u}^{(1)}|1\rangle=0$ by demanding that $w(\bar{y})$ verifies

$$\begin{aligned} \mathcal{L}(w) &= \frac{d}{d\bar{y}}\left[\left(1+2|q|Re\bar{l}^2f'\right)w'\right]=cst\equiv\lambda_w\,, \\ \text{with}\quad w(\bar{y}=0) &= 0\,, \qquad w'(\bar{y}=1)=0\,, \qquad \int_0^1 wd\bar{y}=1 \qquad (12) \end{aligned}$$

We emphasize that, due to the nonlinear (quadratic) nature of the strain-to-stress relation (2d) as implied by the Prandtl's mixing length hypothesis, the appropriate weight w is different from the base flow profile f.

The obtained residual takes the form of an evolution equation for q:

$$\begin{aligned}\varepsilon Re \left[S(q)\partial_t q + F(q)\frac{q}{h}\partial_x q - G(q)\frac{q^2}{h^2}\partial_x h \right] = \\ \frac{Re}{Fr^2} h \left[\sin\theta - \varepsilon \cos\theta \partial_x h + \varepsilon\kappa \partial_{xxx} h \right] - \beta(q)\frac{q}{h^2} \\ +\varepsilon^2 \left[J(q)\frac{q}{h^2}\partial_x h^2 - K(q)\frac{\partial_x q \partial_x h}{h} - L(q)\frac{q}{h}\partial_{xx} h + M(q)\partial_{xx} q \right]\end{aligned} \quad (13)$$

Along with the mass balance (9), (13) forms a closed set of equations governing the evolution of the film, which is formally similar to the classical shallow-water equations (see [6] and next section). Coefficients are function of the local Reynolds number $|q|Re$ only and have been tabulated numerically. Equation (13) being consistent up to $O(\varepsilon)$, the zeroeth order solution to the long-wave expansion, $u^{(0)}$ can be obtained from $\tilde{u}^{(0)}$ where $q = q^{(0)}$ is given by solving $\beta(q)q = (Re/Fr^2)h^3 \sin\theta$.

Results and Validation

Validation of the model (9) and (13) has been conducted with respect to the experimental data by Brock [3]. Brock reported the wave characteristics of roll waves, or periodical hydraulic jumps, that are commonly observed in the torrential regime of free-surface shallow-water hydraulic flows that develop over moderately inclined planes, for inclination angles between $1°$ to $7°$, for Reynolds numbers in the range 4900 to 21000, which corresponds to Froude numbers between 2.63 and 5.90.

In these conditions, surface tension and elongational viscosity are insignificant apart from the hydraulic-jump region that we do not intend to consider. Our averaged momentum equation thus reduces to

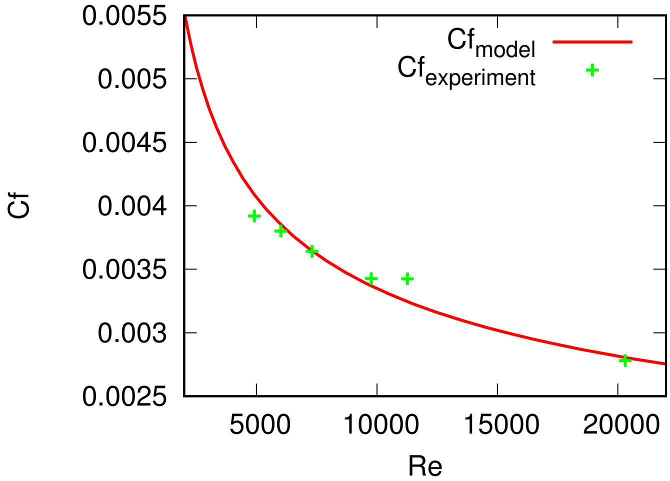
$$\varepsilon Re[S(q)\partial_t q + F(q)\frac{q}{h}\partial_x q - G(q)\frac{q^2}{h^2}\partial_x h] - \frac{Re}{F^2}h[\sin\theta - \varepsilon\cos\theta\partial_x h] + \beta(q)\frac{q}{h^2} = 0\,, \quad (14)$$

which shall be compared to the classical shallow-water momentum equation

$$\varepsilon Re[\partial_t q + 2\frac{q}{h}\partial_x q - \frac{q^2}{h^2}\partial_x h] - \frac{Re}{Fr^2}h[\sin\theta - \varepsilon\cos\theta\partial_x h] + Re C_f \frac{q|q|}{h^2} = 0\,, \quad (15)$$

where the wall shear stress is modelled by the Chezy formula [7]. Comparison of the wall shear stresses in (14) and (15) yields an estimate $C_f \approx \beta(1)/Re$ that agrees well with the measurements as shown in figure 1.

We have used the AUTO07p software (`http://cmvl.cs.concordia.ca/auto/`) to compute the traveling wave solutions to the systems (9), (13) and (9), (15). These solutions are periodical and stationary in a frame of reference $\xi = x - c\,t$ moving at the phase speed c of the waves. They have been obtained by adding a small diffusion term $\propto \partial_x(h\partial_x(q/h))$ to the momentum balances (13) and (15) in order to compute the shock region of the waves. Figures 6 and 7 present the shapes of the computed waves and compare their characteristics (minimum and maximum heights) to Brock's experimental data. In all tested cases, our model yields a better agreement to the experimental data than the corresponding computations with the classical shallow-water equations.



b

Figure 1: C_f versus Re. The continuous line refers to $C_f = \beta(1)/Re$, whereas dots are deduced from the measurements by Brock [3].

h

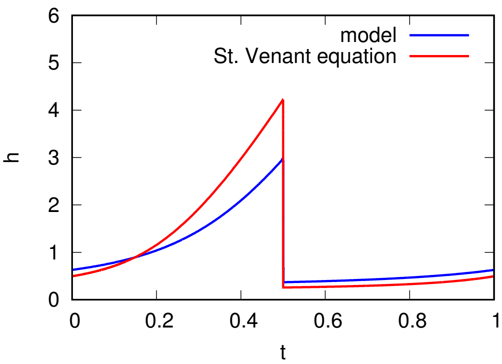


Figure 2: $\lambda = 146, \theta = 6.84, Re = 6011$

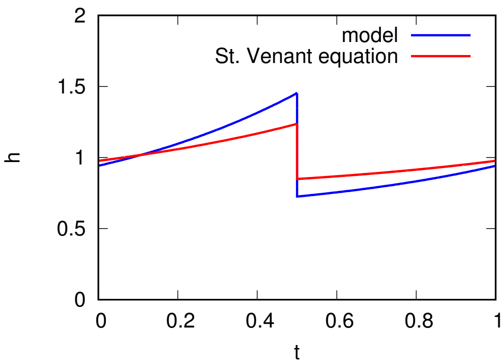


Figure 3: $\lambda = 264, \theta = 1.11, Re = 20310$

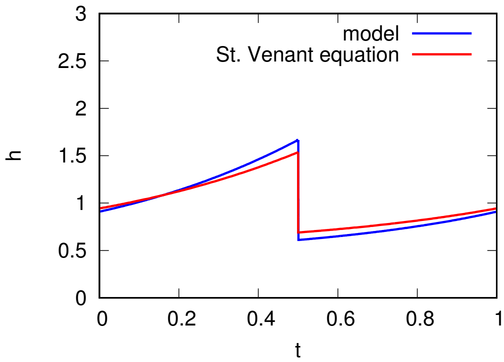


Figure 4: $\lambda = 112, \theta = 2.87, Re = 7302$

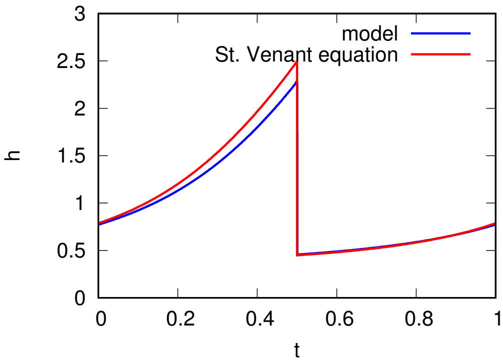


Figure 5: $\lambda = 125, \theta = 4.83, Re = 4914$

Figure 6: Wave profiles

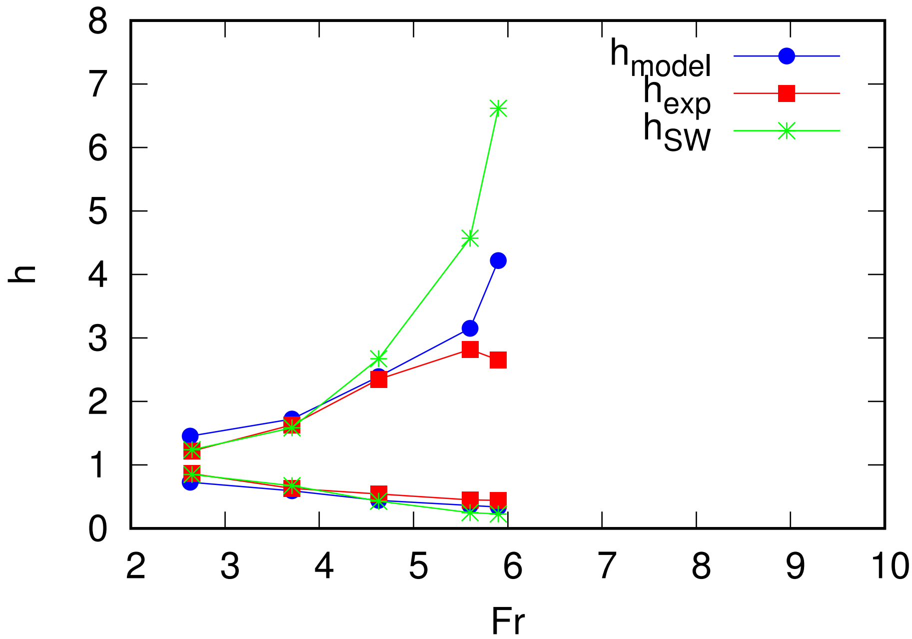


Figure 7: Maximum and minimum heights of the waves corresponding to the experimental data by Brock [3].

Conclusion

We have developed a simplified model to capture the wavy regime of falling liquid films using the weighted-residual technique. This model accounts for the onset of wall-induced turbulence within the framework of the RANS equations with a zero-equation closure. By construction, the derived set of equation is consistent with the long-wave expansion up to $O(\varepsilon)$ for inertial terms and $O(\varepsilon^2)$ for diffusive ones. Comparisons of the travelling-wave solutions to the with Brock's experiments [3] are encouraging. In particular, a better agreement is obtained for the travelling-wave characteristics than with the classical shallow-water equations.

References

[1] Georg F. Dietze, W. Rohlfs, K. Nährich, R. Kneer and D. Scheid, Three-dimensional flow structure in laminar falling liquid films, J. Fluid Mech., 743 (2014), pp.75-123 .

[2] G. L. Richard, C. Ruyer- Quil, and J. P. Vila, A three-equation model for thin films down an inclined plane, J. Fluid Mech.,

[3] Richard R. Brock, A.M. Asce, Periodic permanent roll waves, J. Hudraulics Division, 12 (1970), pp.2565-2580

[4] Philipp Adomeit, Ulrich Renz, Hydrodynamics of three-dimensional waves in laminar falling films, Int. J. Multiphase Flow, 26 (2000), pp.1183-1208

[5] S. Ishigai, S. Nakanisi, T. Koizumi, Z. Oyabu, Hydrodynamics and heat transfer of vertical falling liquid films, Bulletin of the JSME, 15 (1972), pp.594-601.

[6] A.J.C. Barré de Saint-Venant, Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et a l'introduction de marées dans leurs lits, C. Rendus de l'Académie des Sciences, 73 (1871), pp.147-154 and 237-240.

[7] P.G. Holland, Encyclopedia of hydrology and water resource, page- 121, Springer Netherlands, 1998