Bayesian model updating and boundary element method applied to concrete fracture testing.

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Résumé :

Ce papier décrit l'implémentation d'un modèle stochastique pour la mécanique de la rupture. La méthode des éléments de frontière est utilisée pour simuler le comportement mécanique de la structure et des éléments cohésifs permettent de modéliser la propagation de la fissure. De tels éléments sont caractérisés par une loi matériau dédiée, et l'approche probabiliste est utilisée pour prendre en compte les incertitudes associées aux paramètres matériau. Leur densité de probabilité conjointe est identifiée en utilisant l'actualisation bayesienne. Cette méthode est appliquée à un exemple d'illustration qui implique une éprouvette pré-entaillée sollicitée en flexion trois points.

Abstract:

This paper describes an implementation of a stochastic model for fracture mechanics. The boundary element method is used to model the structural response and cohesive zone elements are used to account for the crack propagation. Such elements are characterized by a specific constitutive law and the material parameters are modeled as random variables to take into account the uncertainties. The joint probability density function is identified using Bayesian updating. The method is applied to an application example which involves three point bending tests of a notched concrete specimen.

Keywords: Stochastic Concrete Strength, Bayesian Model Updating, Cohesive Fracture, Boundary Element Method.

1   Introduction

Civil engineers are aware that the fracture behavior of concrete is affected by uncertainties, i.e. identical structures with the same material undergoing the same load exhibit a scatter in their response. This paper describes the implementation of a procedure for the quantification of such uncertainties.

Cohesive zone elements provide an appropriate framework to set up a structural model for fracture mechanics. Such models have been pioneered by Dugdal [1] and Barrenblatt [2]. In this context, fracture is considered as a gradual phenomenon, with the progressive separation of the lips of an extended crack. Cohesive zone elements consist of zero-thickness elements inserted between the bulk material and they account for the resistance to crack opening by the means of a dedicated traction

displacement law. This cohesive force dissipates the energy related to crack formation. The behavior of the bulk material is modeled using the Boundary Element Method (BEM), as this approach provides an appropriate framework for fracture mechanics. Indeed, the shape functions used in BEM approximate accurately the stress concentration in the vicinity of the crack tip and the remeshing procedure during crack growth is simplified by the mesh dimension reduction provided by BEM [3].

In case uncertainties are considered, the parameters of the cohesive crack are identified using inverse methods. Model updating received considerable attention during the past few years [4, 5, 6, 7] and such methods may be used to identify the distribution of the uncertain parameters leading to a good match between the results from the numerical simulations and experimental data. Bayesian statistics [8, 9] are used in this paper to account for uncertainties; it is necessary to formulate statistical hypotheses which are subsequently tested against the experimental observations. The outcome of the Bayesian updating procedure may be interpreted as the plausibility of these hypotheses, conditional the reference data available [10]. This plausibility is quantified using the posterior distribution. It is also possible to compare multiple modeling strategies in order to identify the most suitable strategy. Several methodological developments of Bayesian updating have been performed during the last few years [11, 12, 13], and efficient numerical methods are now available to asses large scale models.

This paper is structured as follows: Section 2 describes the mechanical models; Section 3 outlines the stochastic methods used in this work; a numerical example is proposed in Section 4; conclusions and final remarks are pointed out in Section 5.

2   Boundary element model for fracture mechanics

Fracture mechanics is strongly influenced by the material behavior in the vicinity of the crack tip, in the so-called Fracture Process Zone (FPZ) where large plastic strains are observed. In brittle material, the FPZ is small and linear fracture mechanics is applicable. In ductile material, the FPZ is large and needs to be explicitly accounted for, e.g. using non-linear fracture mechanics. The cohesive elements provide a suitable framework for quasi-brittle material, such as timber, concrete or composite, where a significant FPZ is observed at the crack tip [14].

The cracks are modeled using cohesive zone elements, which consist of one dimension elements inserted at the interface of the traditional bulk elements (for bidimensional models). Such elements describe the cohesive force resisting to the creation of an interface in the material (e.g. a crack) and dissipate the energy related to crack propagation by means of a dedicated traction-displacement law. The use of cohesive elements is a phenomenological approach; multiple formulations of the cohesive law are available and the selection of the best suited may be a challenging task. Three laws are considered for the cohesive elements here; they are shown in Figure 1. In the first cohesive law; an exponential decay of the stress is observed as the displacement is increased, as shown in Figure 1a. It involves two parameters: the ultimate stress of the bulk material f_t and the fracture energy G_t (i.e. the energy required to break the bond between the atoms and create an interface in the material). The second cohesive law is represented in Figure 1b; it involves a bilinear traction displacement curve. This law is as well characterized by the parameters f_t and G_t. A bilinear curve is used in the third law as well; it involves four parameters: f_t and G_t and also two parameters a and b monitoring the location of the change in the slope, as shown in Figure 1c.

The numerical models to be updated are based on the Boundary Element Method (BEM), which is a robust and efficient numerical technique for modeling the behavior of the bulk material in fracture problems. Stress concentrations are represented accurately by the BEM as domain mesh is not required. Moreover, the mesh dimension reduction provided by BEM simplifies the remeshing procedure during crack growth. The BEM has been applied successfully in nonlinear problems over the last decades (see e.g. [3]).
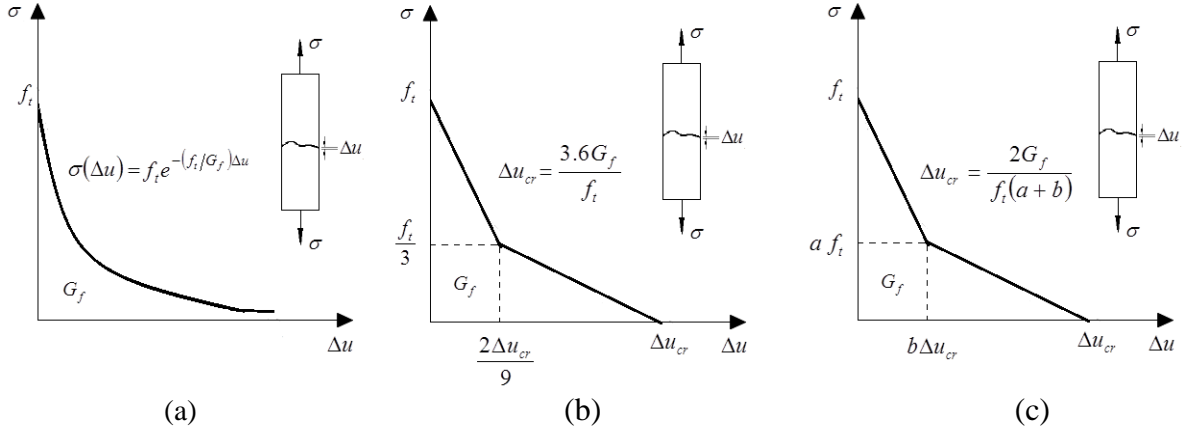
Figure 1. Traction-displacement law associated with the cohesive elements. (a) Exponential law (b) 2-parameter bilinear law; (c) 4-parameter bilinear law.

3   Bayesian updating

The updating procedure proposed in this contribution is based on Bayes' theorem [15], which is formulated as:

$$p(\boldsymbol{\theta}|\mathcal{D},\mathcal{M}) = \frac{p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})p(\boldsymbol{\theta}|\mathcal{M})}{p(\mathcal{D}|\mathcal{M})} \tag{1}$$

where $\boldsymbol{\theta} = \left[\theta_1, \dots, \theta_{N_p}\right]$ is the set of unknown or adjustable parameters, N_p being the total number of parameters; $\mathcal{D}$ denotes the available data (i.e. observations of the response of the investigated structure), and $\mathcal{M}$ denotes the structural model. Four terms are involved in Eq. (1):

1. The *prior distribution* $p(\boldsymbol{\theta}|\mathcal{M})$ is defined without considering the observed data $\mathcal{D}$; this probability density function accounts for the initial knowledge of the uncertain parameters. In this work, this knowledge consists of the upper and lower bound of each parameter. Independent uniform prior distributions between these bounds are considered, which are known as the non-informative priors.

2. The *likelihood* function $p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})$ quantifies the match between the observed data and the model response for the set of uncertain parameters $\boldsymbol{\theta}$. The joint probability density function of the observed data $\mathcal{D}$ is introduced to define the likelihood function.

3. The *posterior distribution* $p(\boldsymbol{\theta}|\mathcal{D},\mathcal{M})$ expresses the knowledge of the uncertain parameters, conditioned by the observed data, the initial knowledge and the structural model. The posterior distribution is used to identify plausible values of the uncertain parameters.

4. The *evidence* $p(\mathcal{D}|\mathcal{M})$ is a normalizing constant guaranteeing that the posterior distribution integrates to one.

Bayesian updating has been successful used in structural engineering [8, 9]; it can be used as an inverse method to identify the distribution of uncertain parameters maximizing the correspondence between experimental data and the outcome of a numerical model [16].

Besides updating the knowledge of the uncertain parameters $\boldsymbol{\theta}$, Bayesian updating may also be used to select the best suited model (see e.g. [17, 18]), in case multiple model classes are considered, i.e. $\mathcal{M} \epsilon [\mathcal{M}_1, \ldots, \mathcal{M}_k]$. First, Bayesian updating is applied for all the possible models. In case all the model classes are assumed to be a priori equally plausible, the likelihood of the model $\mathcal{M}_i$ is proportional to $p(\mathcal{D}|\mathcal{M}_i)$, the evidence term expressed in the denominator of Eq. (1).

The updating procedure also requires an appropriate representation of the posterior distribution. Pointwise estimation of this function may be performed, but this approach does not provide much information. A frequently applied strategy consists of generating samples of the posterior distribution, and the uncertain parameters are subsequently represented as histograms or scatterplots (see e.g.[12]) . The Bayesian Updating with Structural reliability (BUS) method is used here, this strategy allows us to transform the updating problem into a reliability problem. A wide range of algorithms is available to solve reliability problems (see e.g. [19]); subset simulation [20] is used here.

4 Application example

The Bayesian updating procedure is applied to an example of fracture mechanic of concrete; the experimental results are available in the literature [21]. A three point bending test has been performed on a notched specimen; the geometry of the structure is shown in Figure 2a. Five specimens were prepared; and an extensive scatter in the mechanical behavior of the concrete is observed, as shown in Figure 2b.
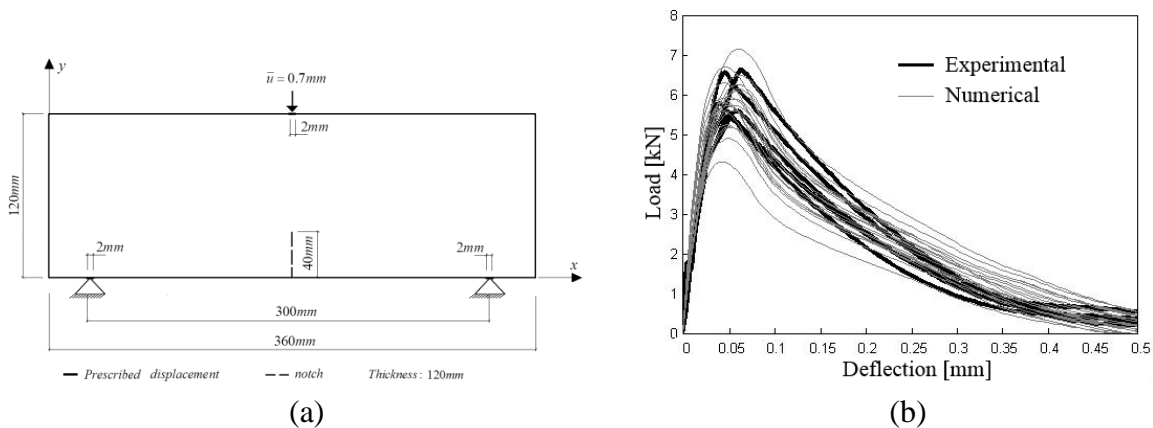
Figure 2. Description of the application example. (a) Geometry of the structure. (b) Comparison of the experimental and numerical results.

The likelihood function is defined using the load deflections curves. The forces associated with the deflection $d = 0.03, 0.05, 0.09$ and 0.25mm are extracted and modeled as a set of random variables with a joint Gaussian probability density function. The moments of the distribution (mean and covariance matrix) are estimated from the experimental data.

Three model classes are considered, with three different formulations of the cohesive law (as shown in Figure 1). The exponential formulation is used in model $\mathcal{M}_1$; the 2-parameter bilinear law is used in $\mathcal{M}_2$ and the 2-parameter bilinear law is used in $\mathcal{M}_3$. The set of uncertain parameters includes the Young modulus of concrete and the parameters of the cohesive law. For model $\mathcal{M}_1$ and $\mathcal{M}_2$, the

tensile strength and the fracture energy are considered; and $\theta = [E, f_t, G_t]$. For model $\mathcal{M}_3$, the x and y-coordinate of the change of the slope of the tensile curve are considered as well, and the vector of the uncertain parameters is $\theta = [E, f_t, G_t, a, b]$.

Table 1 Model classes considered in the Bayesian updating

Model Class	Cohesive law	Random variables
$\mathcal{M}_1$	Exponential	E, f_t, G_t
$\mathcal{M}_2$	Two-parameter bilinear	E, f_t, G_t
$\mathcal{M}_3$	Four-parameter bilinear	E, f_t, G_t, a, b

All the prior distributions are uniform between pre-defined bounds estimated using information available in [21]. The numerical values of the bounds are described in Table 2

Table 2 Bounds of the prior distributions

Random variable	Lower bound	Upper bound	Unit
E	33.28	94.72	GPa
f_t	1.36 ,	6.64	MPa
G_t	100.13	545.87	J/m^2
a	0.1	0.9	-
b	0.1	0.9	-

The Bayesian updating procedure is applied for all the model classes and the evidence (i.e. the normalizing constant associated with Equation (1)) showed that the model $\mathcal{M}_1$ is the most suitable, i.e. the exponential cohesive law lead to the best match between the experimental data and the numerical results. Figure 2b shows the load deflection curves associated with 20 samples drawn from the posterior distribution. These curves are in good agreement with the experimental data.

5   Conclusions

This paper describes a procedure for the identification of the fracture mechanics properties of concrete. The structural behavior is accounted for by means of a boundary element model, and cohesive elements are used to model the crack growth. The uncertainties in the material properties are considered as well and Bayesian updating is used to identify the distribution of the uncertain parameters.
The method is applied to an example involving a notched specimen in three point bending. Three possible classes of models are available; they involve three different formulations of the cohesive law. The method allows us to select the best suited law and to identify the distributions of the uncertain parameters. The numerical results are in good agreement with the experimental data.

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