

Spectral element schemes for the Korteweg-de Vries and Saint-Venant equations

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Résumé :

Les systèmes hyperboliques et les équations dispersives restent difficiles pour les méthodes d'éléments finis (FEMs). Sur la base d'une FEM d'ordre arbitrairement élevé, la méthode des éléments spectraux (SEM), on s'intéresse :

- à l'équation de Korteweg-de Vries, pour montrer comment des dérivées d'ordre élevé peuvent être efficacement traitées par une approximation Galerkin C^0 -continue. On focalise également sur la conservation des invariants, en utilisant notamment des schémas de Runge-Kutta emboîtés.
- aux équations de Saint-Venant 2D, pour montrer comment une SEM stabilisée peut permettre de traiter des chocs. On s'intéresse en particulier à des écoulements avec transition sec-mouillé, en utilisant une variante efficace de la méthode de viscosité entropique.

Abstract :

Hyperbolic systems and dispersive equations remain challenging for finite element methods (FEMs). On the basis of an arbitrarily high order FEM, namely the spectral element method (SEM), we address :

- The Korteweg-de Vries equation, to explain how high order derivative terms can be efficiently handled with a C^0 -continuous Galerkin approximation. The conservation of the invariants is also focused on, especially by using in time embedded implicit-explicit Runge Kutta schemes.
- The 2D shallow water equations, to show how a stabilized SEM can solve problems involving shocks. We especially focus on flows involving dry-wet transitions and propose to this end an efficient variant of the entropy viscosity method.

Key words : Spectral element method, dispersive equations, hyperbolic systems, Korteweg-de Vries equation, shallow water equations, conservation of invariants, entropy viscosity method.

1 Introduction

The Spectral Element Method (SEM) allows a high order approximation of partial differential equations (PDEs) and combines the advantages of spectral methods, that is accuracy and rapid convergence, with those of the finite element method (FEM), that is geometrical flexibility. The SEM is based on a nodal Continuous Galerkin (CG) approach, such that the approximation space contains all C^0 functions

whose restriction in each element is associated to a polynomial of degree N . More precisely, in the master element $(-1, 1)^d$, with d for the space dimension, the basis functions are Lagrange polynomials associated to the $(N + 1)^d$ Gauss-Lobatto-Legendre (GLL) points, which are also used as quadrature points to evaluate the integrals obtained when using a weak form of the problem.

Two academic problems governed by challenging PDEs are considered in this paper :

- The Korteweg-de Vries equation, as a relevant example of dispersive equation ;
- The Saint-Venant equations, as a well known example of non-linear hyperbolic system.

2 SEM approximation of the Korteweg-de Vries (KdV) equation

The KdV problem writes : Find $u(x, t)$, $x \in \Omega \subset \mathbb{R}$ and $t \in \mathbb{R}^+$, such that

$$\partial_t u + u \partial_x u + \beta \partial_{xxx} u = 0$$

where β is a given parameter, and with the initial condition $u(x, t = 0) = u_0(x)$ and e.g. periodic boundary conditions. The main difficulties of this KdV problem are the following :

- Approximation of the dispersive term $\beta \partial_{xxx} u$,
- Preservation of at least two invariants : mass and energy

$$I_1 = \int_{\Omega} u \, dx, \quad I_2 = \int_{\Omega} u^2 \, dx.$$

Approximating the third order derivative term raises a difficulty in the frame of C^0 -CG methods. One may think to address this difficulty with a C^1 -CG method or by using a Petrov-Galerkin approach, but quickly this becomes complex in the multidimensional case or for equations involving higher order dispersive terms. Alternatively, one can introduce an additional variable $f = \partial_{xx} u$, and then address the system that is solved by (u, f) . The number of unknowns is then twice greater, except if the discrete equation that defines f can be trivially solved. This is the case if using the SEM : Because the quadrature and interpolation points coincide, the SEM mass matrix is indeed diagonal so that at the discrete level the intermediate unknown f can be easily eliminated. Discontinuous Galerkin (DG) methods make use of the same trick, but by introducing two intermediate variables $f = \partial_x u$ and $g = \partial_x f$, since DG solutions are not continuous. When using a Local DG (LDG) method the mass matrix is block diagonal, so that its inversion is again not costly. Such an approach can also be used with the SEM, yielding a slightly different definition of the discrete third order differentiation operator, that shows a larger bandwidth but a similar behavior of the condition number with respect to the discretization parameters.

For the approximation in time we suggest using an implicit-explicit (IMEX) scheme, with implicit treatment of the linear third order differentiation term and explicit treatment of the non-linear convection term, so that the numerical stability of the scheme only requires the usual CFL condition. In practice we use the so-called ARS(2,3,3) IMEX scheme, with two implicit steps, three explicit ones and which is third order accurate. When using the SEM the preservation of the mass invariant I_1 is natural, but a difficulty arises with the quadratic invariant I_2 . Two strategies are proposed to overcome this difficulty :

- The first one makes use of embedded IMEX schemes, so that by interpolation between the results obtained at each time step with each of them, one can enforce the conservation of I_2 .

- The second one is more classical, since based on the use of Lagrange multipliers to enforce by L^2 -projection the conservation of I_1 and I_2 .

Both approaches are not specific to the KdV equation or to 1D problems, and so can be used for various PDEs. Finally, it should be mentioned that despite the fact that the KdV equation is not dissipative the SEM approximation is not stabilized. The non-linear transport term is however exactly computed by using for this term an overintegration technique.

Hereafter we consider the KdV equation with $\beta = 0.022^2$ in the periodic domain $(0, 2)$ and assume the initial condition $u_0(x) = \cos(\pi x)$. The numerical solution is computed with $K = 160$ elements, a polynomial approximation degree $N = 5$ and a time step $\tau = 2.5 \cdot 10^{-4}$. The contour levels of the numerical solution in the (x, t) -plane are plotted in Fig. 1 between times 0 and t_R at left, and between times $19t_R$ and $20t_R$ at right, where $t_R \approx 9.68$ is the so called recurrence time, at which one expects to (approximately) recover the initial condition.

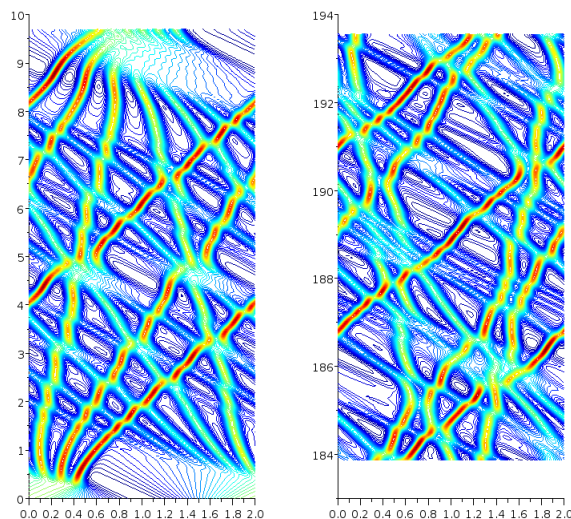


FIGURE 1 – Contour levels of the numerical solution in the (x, t) -plane, between times 0 and t_R at left, between times $19t_R$ and $20t_R$ at right.

All details and references, as well as additional test-cases are provided in [1].

3 Entropy viscosity stabilized SEM for the Saint-Venant system

The Saint-Venant system results from an approximation of the incompressible Euler equations which assumes that the pressure is hydrostatic and that the perturbations of the free surface are small compared to the water height. Then, from the mass and momentum conservation laws and with $\Omega \subset \mathbb{R}^2$ for the computational domain, one obtains equations that describe the evolution of the height $h : \Omega \rightarrow \mathbb{R}^+$ and of the horizontal velocity $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$: For $(\mathbf{x}, t) \in \Omega \times \mathbb{R}^+$:

$$\begin{aligned} \partial_t h + \nabla \cdot (h\mathbf{u}) &= 0 \\ \partial_t (h\mathbf{u}) + \nabla \cdot (h\mathbf{u}\mathbf{u} + gh^2\mathbb{I}/2) + gh\nabla z &= 0 \end{aligned}$$

with \mathbb{I} , identity tensor, $\mathbf{uu} \equiv \mathbf{u} \otimes \mathbf{u}$, g , gravity acceleration, and where $z(\mathbf{x})$ describes the topography, assumed such that $\nabla z \ll 1$. Let us recall the following properties :

- The system is nonlinear and hyperbolic, which means that discontinuities can develop ;
- Assuming that the inlet flow-rate equals the outlet flow-rate, the total mass is preserved ;
- The height h is non-negative ;
- Rest solutions are stable ;
- There exists a convex entropy (actually the energy E) such that

$$\partial_t E + \nabla \cdot ((E + gh^2/2)\mathbf{u}) \leq 0, \quad E = hu^2/2 + gh^2/2 + ghz.$$

Since the solution of the Saint-Venant equations can develop discontinuities even if the initial condition is smooth, its high order SEM approximation should be stabilized. To this end we suggest using a viscous stabilization that preserves the accuracy order of the SEM as long as the solution remains smooth. The entropy viscosity method (EVM) was developed to this end. Essentially, it consists in introducing a viscous term such that

- the transport coefficient is proportional to the absolute value of the residual of the entropy inequality,
- is bounded from above by a first order viscosity ν_{\max} proportional to the wave speed.

The EVM thus relies on strong physical arguments, since away from the shocks the entropy equation is exactly verified. Moreover, the EVM is strongly non-linear, since some viscosity is only introduced when and where a shock develops.

Set $\mathbf{q} = h\mathbf{u}$ and let $h_N(t)$ (resp. $\mathbf{q}_N(t)$) to be the piecewise polynomial continuous approximation of degree N of $h(t)$ (resp. $\mathbf{q}(t)$). In semi-discrete form the proposed stabilized SEM writes : For any w_N, \mathbf{w}_N (scalar and vector valued functions, respectively) spanning the same approximation spaces :

$$\begin{aligned} (\partial_t h_N + \nabla \cdot \mathbf{q}_N, w_N)_N &= -(\nu_h \nabla h_N, \nabla w_N)_N \\ (\partial_t \mathbf{q}_N + \nabla \cdot I_N(\mathbf{q}_N \mathbf{q}_N / h_N) + gh_N \nabla (h_N + z_N), \mathbf{w}_N)_N &= -(\nu_q \nabla \mathbf{q}_N, \nabla \mathbf{w}_N)_N \end{aligned}$$

where $\nu_h \propto \nu_q = \nu$, with ν : entropy viscosity (in the rest of the paper we simply use $\nu_h = \nu_q$). The usual SEM approach is used here : I_N is the piecewise polynomial interpolation operator, based for each element on the tensorial product of Gauss-Lobatto-Legendre (GLL) points, and $(\cdot, \cdot)_N$ stands for the SEM approximation of the $L^2(\Omega)$ inner product, using for each element the GLL quadrature formula which is exact for polynomials of degree less than $2N - 1$ in each variable.

The following remarks may be expressed :

- A stabilization term appears in the mass equation. This is required when a high order approximation is involved, i.e. when the scheme numerical diffusion becomes negligible.
- If the inlet flowrate equals the outlet flow rate and if it is assumed that the Jacobian determinants of the mappings from the reference element $(-1, 1)^2$ to the mesh elements are piecewise polynomials of degree less than N , then the present SEM approximation exactly preserves the mass.
- Thanks to using $\nabla \cdot I_N(gh_N^2 \mathbb{I}/2) \approx gh_N \nabla h_N$ (while h_N^2 is generally piecewise polynomial of degree greater than N), and thus grouping in the momentum equation the pressure and topography terms, a well balanced scheme is obtained by construction : If $\mathbf{q}_N \equiv 0$ and $h_N \neq 0$, then $h_N + z_N = \text{Constant}$.

Another difficulty comes from the positivity of h_N . The algorithm that we propose is the following : In

dry zones, i.e. for any element Q_{dry} such that at one GLL point $\min h_N < h_{thresh}$, where h_{thresh} is a user defined threshold value (typically a thousandth of the reference height) :

- Modify the entropy viscosity technique, by using in Q_{dry} the upper bound first order viscosity :

$$\nu = \nu_{max} \quad \text{in } Q_{dry}$$

- In the momentum equation assume that :

$$h_N g \nabla (h_N + z_N) \equiv 0 \quad \text{in } Q_{dry}$$

Hereafter, for the discretization in time we simple use the standard forth order RK4 scheme.

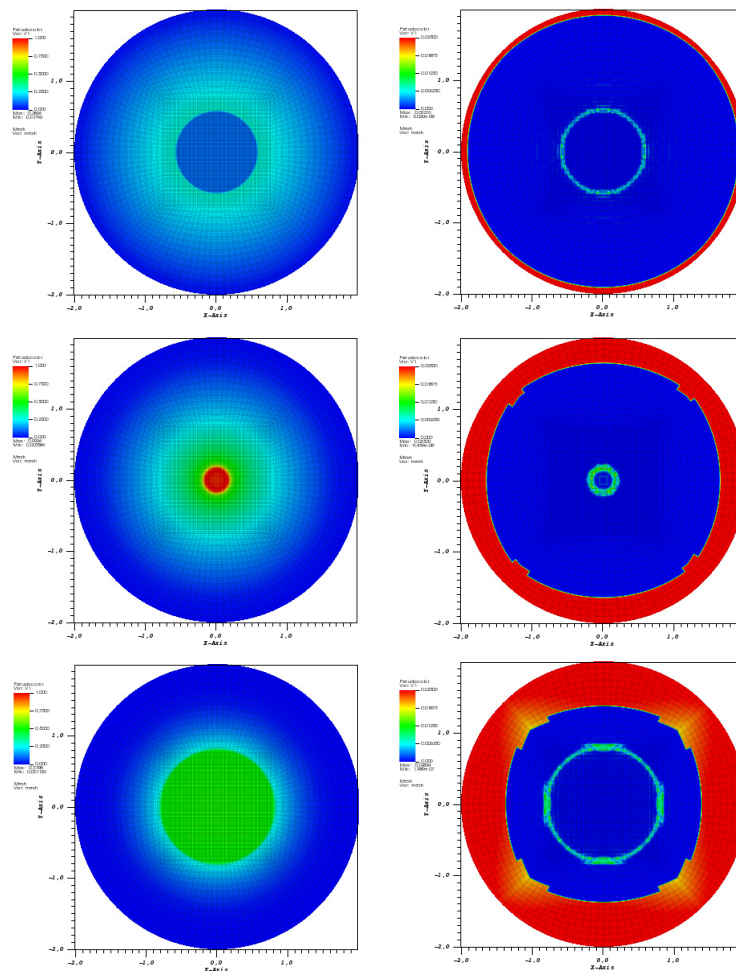


FIGURE 2 – Axisymmetric oscillations with shocks in a paraboloid : Height (left, in $[0, 1]$) and entropy viscosity (right, in $[0, 0.025]$), at $t \approx 1.4$ (top), $t \approx 1.65$ (middle) and $t \approx 1.9$ (bottom).

In Fig. 2 we give the results of a simulation that shows axisymmetric oscillations in the paraboloid $z(\mathbf{x}) = 0.1\mathbf{x}^2$, $|\mathbf{x}| < 2$, and that involves both dry-wet transitions and shocks. The initial condition is the following :

$$h(t = 0) \equiv h_0 = \max(1 - \mathbf{x}^2, 0), \quad \mathbf{u} = (0, 0),$$

and at the boundary an impermeability condition together with an homogeneous Neumann condition

for h are imposed. The discretization parameters are the following, number of elements : 2352, polynomial approximation degree : $N = 5$, resulting number of grid-points : 59081. Calculations have been made till time $t = 5$ with time-step 10^{-4} . We use $h_{thresh} = \max_{\Omega} h_0/500$ and at the initial time $h_N = I_N(\max(h_0, h_{thresh}/2))$. Such a flow is alternatively expanding and then retracting towards the paraboloid axis. Fig. 2 shows the flow at three different times, during the first retraction-expansion phase : At $t \approx 1.4$ the velocity field is inwards, at $t \approx 1.65$ it is close to reversal and at $t \approx 1.9$ it is outwards. The height h_N (at left) and the entropy viscosity ν (at right) are visualized. As desired, the entropy viscosity saturates in dry zones and also focuses at the shock.

Details and references are given in [2].

Références

- [1] S. Minjeaud, R. Pasquetti, High order C^0 -continuous Galerkin schemes for high order PDEs, conservation of quadratic invariants and application to the Korteweg-de Vries model, *J. Sci. Comp.*, in press.
- [2] R. Pasquetti, Viscous stabilizations for high order approximations of Saint-Venant and Boussinesq flows, *Lecture Notes in Computational Science and Engineering : Spectral and High Order Methods for Partial Differential Equations-ICOSAHOM 2016*, Springer, in press.